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UNIQUENESS OF COXETER STRUCTURES ON KAC–MOODY ALGEBRAS

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ABSTRACT. Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra, and $U_\hbar\mathfrak{g}$ the corresponding quantum group. We showed in [1, 2] that the braided Coxeter structure on integrable, category $\mathcal{O}$ representations of $U_\hbar\mathfrak{g}$ which underlies the R–matrix actions arising from the Levi subalgebras of $U_\hbar\mathfrak{g}$ and the quantum Weyl group action of the generalised braid group $B_\mathfrak{g}$ can be transferred to integrable, category $\mathcal{O}$ representations of $\mathfrak{g}$. We prove in this paper that, up to unique equivalence, there is a unique such structure on the latter category with prescribed restriction functors, R–matrices, and local monodromies. This extends, simplifies and strengthens a similar result of the second author valid when $\mathfrak{g}$ is semisimple, and is used in [3] to describe the monodromy of the rational Casimir connection of $\mathfrak{g}$ in terms of the quantum Weyl group operators of $U_\hbar\mathfrak{g}$. Our main tool is a refinement of Enriquez's universal algebras, which is adapted to the PROP describing a Lie bialgebra graded by the non–negative roots of $\mathfrak{g}$.

CONTENTS

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1. Introduction

1.1. This is the second of three papers whose goal is to extend the description of the monodromy of the rational Casimir connection of a semisimple Lie algebra in terms of quantum Weyl groups given in [25, 26, 27, 28] to the case of an arbitrary symmetrisable Kac–Moody algebra $\mathfrak{g}$.

In [2], we introduced the notion of braided Coxeter category, which is informally a tensor category carrying commuting actions of Artin's braid groups and a given generalised braid group on the tensor product of its objects. We showed that such a structure arises from the quantum group $U_\hbar\mathfrak{g}$, specifically on the category $\mathcal{O}_\hbar^{\mathrm{int}}$ of integrable, highest weight representations of $U_\hbar\mathfrak{g}$. The corresponding Artin group actions are given by the universal R–matrices of the Levi subalgebras of $U_\hbar\mathfrak{g}$, and the action of the generalised braid group of $\mathfrak{g}$ by the quantum Weyl group operators of $U_\hbar\mathfrak{g}$. The main result of [2] is that this structure can be transferred to the category $\mathcal{O}^{\mathrm{int}}$ of integrable, highest weight modules for $\mathfrak{g}$. The transfer relies on a 2–categorical, relative version of Etingof–Kazhdan quantisation, which takes as input a split inclusion of Lie bialgebras $\mathfrak{a} \subset \mathfrak{b}$, and allows to construct an equivalence $\mathcal{O}_\hbar^{\mathrm{int}} \cong \mathcal{O}^{\mathrm{int}}$ which is compatible with a given chain of Levi subalgebras of $\mathfrak{g}$ [1].

1.2. The goal of the present paper is to prove that $\mathcal{O}^{\mathrm{int}}$ possesses, up to unique equivalence, a unique braided Coxeter structure with prescribed restrictions functors, R–matrices and local monodromies. This is used in [3] to prove that the monodromy of the rational Casimir connection of $\mathfrak{g}$ is described by the quantum Weyl group operators of $U_\hbar\mathfrak{g}$, by showing that the monodromy of the rational KZ and Casimir connections arise from a braided quasi–Coxeter structure on $\mathcal{O}^{\mathrm{int}}$.[1]

1.3. The uniqueness of braided Coxeter structures on $\mathcal{O}^{\mathrm{int}}$ is obtained from a cohomological rigidity result, as is the case for a semisimple Lie algebra. The proof of this result, however, differs significantly from that given in [26, 27]. Indeed, the latter relies on the well–known computation of the Hochschild (coalgebra) cohomology of the enveloping algebra $U\mathfrak{g}$ in terms of the exterior algebra of $\mathfrak{g}$. For an arbitrary Kac–Moody algebra, the tensor powers of $U\mathfrak{g}$ need to be replaced by their completion $U\mathfrak{g}_{\mathcal{O}}^{\otimes n}$ with respect to category $\mathcal{O}$. Indeed, $U\mathfrak{g}$ and $U\mathfrak{g}^{\otimes 2}$ do not contain the Casimir operator C of $\mathfrak{g}$ and the invariant tensor $2\Omega = \Delta(C) - C\otimes 1 - 1\otimes C$ respectively, and are therefore not appropriate receptacles for the coefficients of the Casimir and KZ connections. While the computation of the Hochschild cohomology of $U\mathfrak{g}$ holds for an arbitrary Lie algebra, it is not known to do so, and may in fact fail, for the topological coalgebra $U\mathfrak{g}_{\mathcal{O}}$, which seems to have a rather unwieldy cohomology.

1.4. Rather than using the completions $U\mathfrak{g}_{\mathcal{O}}^{\otimes n}$, we rely on a refinement of Enriquez's universal algebras $U\mathfrak{G}^n_{\mathrm{univ}}$ [10]. These arise from the PROP of Lie bialgebras, and were used by Enriquez to give a cohomological construction of quantisation functors for Lie bialgebras [12]. They are universal in the following sense: for any Lie bialgebra $\mathfrak{b}$ with Drinfeld double $\mathfrak{g}_\mathfrak{b} = \mathfrak{b} \oplus \mathfrak{b}^*$ and any $n \geq 1$, $U\mathfrak{G}^n_{\mathrm{univ}}$ maps

[1]More precisely, for an arbitrary symmetrisable Kac–Moody algebra only the normally ordered version of the Casimir connection introduced in [18] can be defined. We show in [3], however, that its monodromy can be modified by a cocyle so as to become equivariant under the Weyl group, and that the resulting action of the braid group action is described by the quantum Weyl group operators of $U_\hbar\mathfrak{g}$.

to a completion $\widehat{U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}}$ of the n–fold tensor product of the enveloping algebra of $\mathfrak{g}_{\mathfrak{b}}$. For $n=1$, the image of $U\mathfrak{G}^n_{\text{univ}}$ in $\widehat{U\mathfrak{g}_{\mathfrak{b}}}$ is the subalgebra spanned by the interlaced powers of the normally ordered Casimir operator of $\mathfrak{g}_{\mathfrak{b}}$, *i.e.,* the elements

$$\kappa_N^\sigma = \sum_{i_1,\ldots,i_N} b_{i_1} b_{i_2}\cdots b_{i_N}\cdot b^{i_{\sigma(N)}} b^{i_{\sigma(N-1)}}\cdots b^{i_{\sigma(1)}} \tag{1.1}$$

where $\{b_i\},\{b^i\}$ are dual bases of $\mathfrak{b}$ and $\mathfrak{b}^*$, N is an arbitrary integer, and σ a permutation in $\mathfrak{S}_N$.

The completion $\widehat{U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}}$ is with respect to the category $\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}$ of *equicontinuous* $\mathfrak{g}_{\mathfrak{b}}$–modules, which are those on which the action of $\mathfrak{b}^*$ (and therefore the sum (1.1)) is locally finite. If $\mathfrak{g}$ is a symmetrisable Kac–Moody algebra with negative Borel subalgebra $\mathfrak{b}$, the realisation of $\mathfrak{g}$ as a quotient of the Drinfeld double of $\mathfrak{b}$ gives rise to an embedding of category $\mathcal{O}$ for $\mathfrak{g}$ as a full subcategory of $\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}$.

The coproduct on $U\mathfrak{g}_{\mathfrak{b}}$ gives rise to a cosimplicial structure on the tower of algebras $\{\widehat{U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}}\}_{n\geqslant 0}$. The latter can be lifted to $\{U\mathfrak{G}^n_{\text{univ}}\}_{n\geqslant 0}$, and gives rise to a Hochschild complex. Enriquez's crucial insight is that this complex contains enough elements to allow for the construction of quantisation functors, yet has a manageable cohomology, which is given by a universal version of the exterior algebra of $\mathfrak{g}_{\mathfrak{b}}$.

1.5. We give in this paper an alternative, and perhaps more natural construction of $U\mathfrak{G}^n_{\text{univ}}$ by using Drinfeld–Yetter modules over a Lie bialgebra $\mathfrak{b}$. Such a module is a triple (V,π,π^*) where $\pi:\mathfrak{b}\otimes V\to V$ gives V the structure of a left $\mathfrak{b}$–module, $\pi^*: V\to\mathfrak{b}\otimes V$ that of a right $\mathfrak{b}$–comodule, and π,π^* satisfy a compatibility condition [15]. The latter is designed so as to give rise to an action of the Drinfeld double of $\mathfrak{b}$, with $\phi\in\mathfrak{b}^*\subset\mathfrak{g}_{\mathfrak{b}}$ acting on V by $\phi\otimes\mathrm{id}_V\circ\pi^*$.

The symmetric tensor category $\mathsf{DY}_{\mathfrak{b}}$ of such modules coincides with that of equicontinuous $\mathfrak{g}_{\mathfrak{b}}$–modules, with the coaction of $\mathfrak{b}$ on $V\in\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}$ given by $\pi^*(v)=\sum_i b_i\otimes b^i\, v$ [14]. Under this correspondence, the action of the normally ordered Casimir $\kappa=\sum_i b_ib^i$ of $\mathfrak{g}_{\mathfrak{b}}$ on $V\in\mathsf{DY}_{\mathfrak{b}}$ is simply given by $\pi\circ\pi^*$. More generally, the interlaced Casimir κ_N^σ (1.1) acts on V by the composition of the iterated coaction $(\pi^*)^{(N)}: V\to\mathfrak{b}^{\otimes N}\otimes V$ followed by the permutation $\sigma^{-1}\otimes\mathrm{id}_V$ and the iterated action $\pi^{(N)}:\mathfrak{b}^{\otimes N}\otimes V\to V$. Similarly, the r–matrix of $\mathfrak{g}_{\mathfrak{b}}$ given by $r=\sum_i b_i\otimes b^i\in\mathfrak{b}\widehat{\otimes}\mathfrak{b}^*$ acts on a tensor product $V\otimes W$ as the composition

$$r_{VW}=\pi_V\otimes\mathrm{id}_W\circ(1\,2)\circ\mathrm{id}_V\otimes\pi_W^*$$

For any $n\geq 1$, we introduce a colored PROP $\underline{\mathsf{DY}}^n$ which describes a Lie bialgebra $\mathfrak{b}$, together with n Drinfeld–Yetter modules $V_1,\ldots,V_n$ over $\mathfrak{b}$. We then consider the algebra $\mathfrak{U}^n_{\mathsf{DY}}=\mathrm{End}_{\underline{\mathsf{DY}}^n}(V_1\otimes\cdots\otimes V_n)$, and show it to be isomorphic to Enriquez's algebra $U\mathfrak{G}^n_{\text{univ}}$. This alternative construction makes the algebra structure on $U\mathfrak{G}^n_{\text{univ}}$, and its action on equicontinuous $\mathfrak{g}_{\mathfrak{b}}$–modules far more transparent.

1.6. We then introduce three refinements of the algebras $\mathfrak{U}^n_{\mathsf{DY}}$. The first one, $\mathfrak{U}^n_{\mathsf{PDY}}$, is obtained from the colored PROP describing a split inclusion of Lie bialgebras $\mathfrak{a}\subset\mathfrak{b}$, together with n Drinfeld–Yetter modules over $\mathfrak{b}$. The image of $\mathfrak{U}^1_{\mathsf{PDY}}$ in $\widehat{U\mathfrak{g}_{\mathfrak{b}}}$ is spanned by the interlaced products of the normally ordered Casimir operators of the doubles of $\mathfrak{a}$ and $\mathfrak{b}$. The Hochschild cohomology of the tower $\mathfrak{U}^n_{\mathsf{PDY}}$ can be computed via the calculus of Schur functors developed in [12], and shows in particular that the relative quantisation functor constructed in [1] is unique up to unique isomorphism.[2]

The second refinement, $\mathfrak{U}^n_{\mathsf{S}}$, is obtained in a similar way from a PROP describing n Drinfeld–Yetter modules over a Lie bialgebra $\mathfrak{b}$ which is graded by a partial abelian semigroup S. The image of $\mathfrak{U}^n_{\mathsf{S}}$ in $\widehat{U\mathfrak{g}_\mathfrak{b}}$ is then spanned by the interlaced products of the normally ordered Casimir operators of the subspaces $\mathfrak{b}_\alpha \oplus \mathfrak{b}^*_\alpha \subset \mathfrak{g}_\mathfrak{b}$, $\alpha \in \mathsf{S}$. When S is the partial semigroup $\mathsf{R}_+ \sqcup \{0\}$ consisting of the positive roots of a symmetrisable Kac–Moody algebra $\mathfrak{g}$ together with zero, this makes $\mathfrak{U}_{\mathsf{S}}$ an appropriate receptacle for the coefficients of the Casimir connection of $\mathfrak{g}$.

The third refinement, $\mathfrak{U}^n_{\mathbb{S}}$, is prompted by the following. We show in [2] that the braided (pre–)Coxeter structure transferred from $\mathcal{O}^{\mathrm{int}}_\hbar$ to $\mathcal{O}^{\mathrm{int}}$ is *diagrammatic*, *i.e.,* compatible with the Lie subalgebras $\mathfrak{g}_B$ generated by the root vectors $\{e_i, f_i\}_{i\in B}$ corresponding to a subdiagram B of the Dynkin diagram of $\mathfrak{g}$. In particular, this structure cannot be lifted to a braided (pre–)Coxeter structure on $\mathfrak{U}^\bullet_{\mathsf{S}}$, since the latter only accounts for the Cartan subalgebra of $\mathfrak{g}$ and not its subspaces $\mathfrak{h}_B$ spanned by $\{\alpha_i^\vee\}_{i\in B}$.

The definition of $\mathfrak{U}^n_{\mathbb{S}}$ relies on a *diagrammatic semigroup* $\mathbb{S}$, and accounts for both the root space decomposition of $\mathfrak{g}$ as well as for its diagrammatic subalgebras $\mathfrak{g}_B$. The braided (pre–)Coxeter structures coming from the quantum group and the Casimir connection can then both be realized in $\mathfrak{U}^\bullet_{\mathbb{S}}$, as we show in [2] and [3], respectively. The computation of the Hochschild cohomology of $\mathfrak{U}^n_{\mathbb{S}}$ yields the required rigidity result, thus allowing to prove they are isomorphic.

1.7. The use of the algebras $\mathfrak{U}^n_{\mathbb{S}}$ leads to far stronger uniqueness results than had been obtained in [26, 27] for a semisimple Lie algebra $\mathfrak{g}$. Indeed, as is the case for the universal algebras $U\mathfrak{G}^n_{\mathrm{univ}}$, the tower $\{\mathfrak{U}^n_{\mathbb{S}}\}_{n\geq 0}$ has trivial first Hochschild cohomology, which implies that the isomorphism of two braided, Coxeter structures is unique up to a *unique* gauge. This raises the hope that the equivalences we construct may be convergent as series in the deformation parameter $\hbar$, and could in particular be specialised to numerical, non–rational, values of $\hbar$. It is also worth pointing out that the vanishing of the first Hochschild cohomology removes the need for the use of Dynkin diagram cohomology developed in [27] to deal with secondary obstructions, thereby simplifying the proof of rigidity even for a semisimple Lie algebra.

1.8. We now review our results in more detail. A PROP is a categorical realisation of an algebraic structure. More precisely, given a field k of characteristic zero, a PROduct–Permutation category is a k–linear, symmetric monoidal category with objects the non–negative numbers and tensor product $[n]\otimes[m] = [n+m]$. For example, the PROP $\underline{\mathsf{LA}}$ of Lie algebras is generated by an anti–symmetric morphism $\mu : [2] \to [1]$ satisfying the Jacobi identity. One can then think of Lie algebras over k as symmetric monoidal functors from $\underline{\mathsf{LA}}$ to k–vector spaces, and morphisms of Lie algebras as natural transformations of the corresponding realisation functors.

1.9. Richer structures can be described by *colored* PROPs, whose objects are sequences in a given set of colors A. A key example for us is the PROP $\underline{\mathsf{DY}}^n$ on

[2]The uniqueness of the isomorphism follows from the fact that the first Hochschild cohomology of $\mathfrak{U}^\bullet_{\mathsf{PDY}}$ is zero, as is the case for $U\mathfrak{G}^\bullet_{\mathrm{univ}}$. Thus, figuratively speaking, $U\mathfrak{G}^\bullet_{\mathrm{univ}}$ and $\mathfrak{U}^\bullet_{\mathsf{PDY}}$ behave like the tensor powers of an enveloping algebra without primitive elements.

$n+1$ colors which we introduce in Section 5. In this case, the category of symmetric monoidal functors $\underline{\mathsf{DY}}^n \to \mathrm{Vect}_{\mathsf{k}}$ is isomorphic to that of tuples $(\mathfrak{b}; V_1, \ldots, V_n)$ consisting of a Lie bialgebra $\mathfrak{b}$ over k, and n Drinfeld–Yetter modules $V_1, \ldots, V_n$.

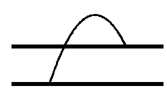
A natural transformation of these functors amounts to a tuple $(\phi; f_1, \ldots, f_n)$, where $\phi : \mathfrak{b} \to \mathfrak{c}$ is a morphism of Lie bialgebras, and each $f_i : V_i \to W_i$ is both a morphism of $\mathfrak{b}$–modules $V_i \to \phi^* W_i$ and of $\mathfrak{c}$–comodules $\phi_* V_i \to W_i$. In particular, choosing $\phi = \mathrm{id}$ shows that any endomorphism of $V_1 \otimes \cdots \otimes V_n$ in $\underline{\mathsf{DY}}^n$ commutes with morphisms of Drinfeld–Yetter modules over any Lie bialgebra $\mathfrak{b}$. Thus, if $\mathsf{f} : \mathsf{DY}_{\mathfrak{b}} \to \mathrm{Vect}_{\mathsf{k}}$ is the forgetful functor, the algebra

$$\mathfrak{U}^n_{\mathsf{DY}} = \mathsf{End}_{\underline{\mathsf{DY}}^n}(V_1 \otimes \cdots \otimes V_n)$$

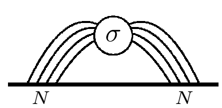
maps to the endomorphisms of $\mathsf{f}^{\otimes n}$, and therefore to the completion of $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}$ with respect to equicontinuous modules.

1.10. The category $\underline{\mathsf{DY}}^n$ is best described diagrammatically. The identity on the universal Lie bialgebra (resp. Drinfeld–Yetter module) is represented by a thin (resp. bold) horizontal line, and the bracket, cobracket, action and coaction by the diagrams

which are read from left to right. By 1.5, the normally ordered Casimir $\kappa \in \mathfrak{U}^1_{\mathsf{DY}}$ and r–matrix $r \in \mathfrak{U}^2_{\mathsf{DY}}$ are therefore represented, respectively, by

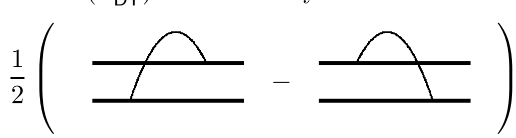
 and 

In Sections 4 and 5, we explicitly describe the morphisms in $\underline{\mathsf{DY}}^n$, and construct an integral basis for $\mathfrak{U}^n_{\mathsf{DY}}$ which, for $n = 1$ is given by the diagrams



where $N \geqslant 0$ and $\sigma \in \mathfrak{S}_N$. By 1.5, these correspond to the interlaced powers of the normally ordered Casimir (1.1). This description leads to a PBW theorem for $\mathfrak{U}^n_{\mathsf{DY}}$, and the explicit computation of its Hochschild cohomology, which is analogous to the fact that $H^n(U\mathfrak{g}_{\mathfrak{b}}) = \wedge^n \mathfrak{g}_{\mathfrak{b}}$.

Pictorially, an element in $H^n(\mathfrak{U}^{\bullet}_{\mathsf{DY}})$ is a linear combination of anti–symmetric diagrams with n bold lines, and exactly one action or one coaction on each of these. For example, $H^1(\mathfrak{U}^{\bullet}_{\mathsf{DY}}) = 0$ (*i.e.,* $\mathfrak{U}^1_{\mathsf{DY}}$ has no primitive elements), and the simplest non–trivial element in $H^2(\mathfrak{U}^{\bullet}_{\mathsf{DY}})$ is the anti–symmetric r–matrix



1.11. The algebra $\mathfrak{U}^n_{\mathsf{DY}}$ is a universal receptacle for the coefficient of the KZ connection on n points for a Drinfeld double $\mathfrak{g}_{\mathfrak{b}}$ since, for $n = 2$, it contains the invariant tensor $\Omega = \sum_i b_i \otimes b^i + b^i \otimes b_i = r + r_{21}$. However, $\mathfrak{U}^1_{\mathsf{DY}}$ is too small to contain the coefficients of the Casimir connection of a symmetrisable Kac–Moody algebra

$\mathfrak{g}$, since it does not account for the root space decomposition of $\mathfrak{g}$, and in particular the diagrammatic and Levi subalgebras

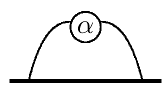
$$\mathfrak{g}_B = \langle e_i, f_i \rangle_{i\in B} \subset \mathfrak{l}_B = \mathfrak{g}_B + \mathfrak{h} \subset \mathfrak{g}$$

corresponding to a subdiagram B of the Dynkin diagram of $\mathfrak{g}$.

To this end, we first introduce and study in Section 6 the PROP $\underline{\mathsf{PDY}}^n$ obtained by adding to $\underline{\mathsf{DY}}^n$ an idempotent endomorphism of the universal Lie bialgebra. Its image is then a split Lie subbialgebra. The results of Section 5, in particular the PBW Theorem and computation of Hochschild cohomology extend easily to the algebras $\mathfrak{U}^n_{\mathsf{PDY}} = \mathsf{End}_{\underline{\mathsf{PDY}}^n}(V_1 \otimes \cdots \otimes V_n)$.

From here, the PROPic construction of Levi subalgebras is fairly straightforward. We first observe that the negative Borel of a symmetrisable Kac–Moody algebra $\mathfrak{g}$ is graded, as a Lie bialgebra, by the partial semigroup R_0 consisting of the positive roots of $\mathfrak{g}$ and zero. The PROP encoding this structure is denoted $\underline{\mathsf{DY}}^n_{\mathsf{S}}$, where S is any partial abelian semigroup. It is obtained from $\underline{\mathsf{DY}}^n$ by adding a complete family of orthogonal idempotents labelled by the elements of S. In the case of the semigroup R_0, by considering the sum of the idempotents corresponding to zero or a root associated to a subdiagram B of the Dynkin diagram, one obtains a universal analogue of the Levi subalgebra $\mathfrak{l}_B$.

The universal algebra $\mathfrak{U}^n_{\mathsf{S}} = \mathsf{End}_{\underline{\mathsf{DY}}^n_{\mathsf{S}}}(V_1 \otimes \cdots \otimes V_n)$ is generated by arc diagrams, in which each thin line is now labeled by S. In particular, $\mathfrak{U}^1_{\mathsf{S}}$ contains the elements



for any $\alpha \in \mathsf{S}$. In the case of the semigroup of non–negative roots R_0, these diagrams correspond precisely to the normally ordered Casimir elements of the $\mathfrak{sl}_2$–triple of the root α, and make $\mathfrak{U}^1_{\mathsf{S}}$ a universal receptacle for the coefficients of the Casimir connection.

The universal algebras $\mathfrak{U}^n_{\mathsf{S}}$, however, do not provide a universal realization of the diagrammatic subalgebras $\mathfrak{g}_B$, which are necessary to describe the braided (pre–)Coxeter structure transferred from $\mathcal{O}^{\mathrm{int}}_{\hbar}$. We therefore introduce a refinement $\underline{\mathsf{DY}}^n_{\mathbb{S}}$ of the PROP $\underline{\mathsf{DY}}^n_{\mathsf{S}}$, where we further decompose the idempotent corresponding to the zero element of S, so as to reproduce the subspaces $\mathfrak{h}_B = \langle \alpha_i^\vee \rangle_{i\in B} \subset \mathfrak{h}$. The corresponding universal algebras $\mathfrak{U}^n_{\mathbb{S}}$ account for both the root space decomposition of $\mathfrak{g}$, and therefore the coefficients of the Casimir connection, as well as for its diagrammatic subalgebras.

1.12. We now sketch the definition of a braided Coxeter category. We refer to an unoriented graph D with no multiple edges or loops as a diagram, and to its full subgraphs $B \subseteq D$ as subdiagrams. A braided *pre*–Coxeter category $\mathcal{Q}$ of type D consists of the following three pieces of data

- **Diagrammatic categories.** For any subdiagram $B \subseteq D$, a braided tensor category $\mathcal{Q}_B$.
- **Restriction functors.** For any pair of subdiagrams $B' \subseteq B$, a (not necessarily braided) monoidal functor $F^{\mathcal{F}}_{B'B} : \mathcal{Q}_B \to \mathcal{Q}_{B'}$ depending upon the choice of a maximal chain of subdiagrams $B = B_1 \supsetneq B_2 \supsetneq \cdots \supsetneq B_m = B'$.

- **Associators.** For any $B' \subseteq B$, and pair of maximal chains $\mathcal{G}, \mathcal{F}$ from B to B', an isomorphism of monoidal functors $\Upsilon^{\mathcal{G}\mathcal{F}} : F^{\mathcal{F}}_{B'B} \to F^{\mathcal{G}}_{B'B}$.[3]

The above data satisfies various requirements. In particular, the restriction functors and associators are compatible with the composition of chains corresponding to triple inclusions $B'' \subset B' \subset B$ and, for any $B' \subset B$ and maximal chains $\mathcal{F}, \mathcal{G}, \mathcal{H}$ from B to B', one has $\Upsilon^{\mathcal{H}\mathcal{G}} \circ \Upsilon^{\mathcal{G}\mathcal{F}} = \Upsilon^{\mathcal{H}\mathcal{F}}$.

$\mathcal{Q}$ is a braided Coxeter category if it is further endowed with distinguished elements $S_i^{\mathcal{Q}} \in \mathsf{Aut}(F_i)$ where i ranges over the vertices of D, which satisfy the following version of the braid relations determined labeling the edges of D by multiplicities $m_{ij} = m_{ji} \in \{2, 3, \dots, \infty\}$. For any $i \neq j$ such that $m_{ij} < \infty$, and maximal chains $\mathcal{F}, \mathcal{G}$ from D to the empty subdiagram such that $\mathcal{F}$ (resp. $\mathcal{G}$) contains i (resp. j) among the connected components of its elements, the following holds in $\mathrm{Aut}(F^{\mathcal{F}}_{\emptyset D})$

$$\underbrace{\mathsf{Ad}\left(\Upsilon^{\mathcal{G}\mathcal{F}}\right)(S_j^{\mathcal{Q}}) \cdot S_i^{\mathcal{Q}} \cdot \mathsf{Ad}\left(\Upsilon^{\mathcal{G}\mathcal{F}}\right)(S_j^{\mathcal{Q}}) \cdots}_{m_{ij}} = \underbrace{S_i^{\mathcal{Q}} \cdot \mathsf{Ad}\left(\Upsilon^{\mathcal{G}\mathcal{F}}\right)(S_j^{\mathcal{Q}}) \cdot S_i^{\mathcal{Q}} \cdots}_{m_{ij}}$$

This gives rise to an action $\rho^{\mathcal{F}} : B_D \to \mathrm{Aut}(F^{\mathcal{F}}_{\emptyset D})$ of the Artin braid group B_D determined by the labeling of D, which is intertwined by the associators $\Upsilon^{\mathcal{G}\mathcal{F}}$.

1.13. Let $\mathfrak{g}$ be the symmetrisable Kac–Moody algebra with Dynkin diagram D. For any subdiagram $B \subseteq D$, we denote by $\mathfrak{g}_B \subseteq \mathfrak{g}$ the diagrammatic subalgebra and by $\mathfrak{b}_B \subseteq \mathfrak{b}$ its negative Borel subalgebra. To study braided pre–Coxeter structure on Drinfeld–Yetter modules over $\{\mathfrak{b}_B\}_{B \subseteq D}$, $\mathfrak{b}$ has to be *diagrammatic*, *i.e.,* for any $B' \subseteq B$, $\mathfrak{b}_{B'} \subseteq \mathfrak{b}_B$ and, for any $B' \perp B$, $[\mathfrak{b}_{B'}, \mathfrak{b}_B] = 0$. [4] Although Kac–Moody algebras of finite, affine, or hyperbolic type are diagrammatic, not all are diagrammatic with counterexamples already in rank 4 (cf.[2]). To remedy this, in Section 15, we consider certain split central extensions, referred to as *extended Kac–Moody algebras*, whose Borel subalgebras are canonically endowed with a split diagrammatic structure.

Let $\overline{\mathfrak{g}}$ be a diagrammatic or extended symmetrisable Kac–Moody algebras with diagrammatic semigroup of positive roots $\mathbb{S}$ and universal algebras $\mathfrak{U}^n_{\mathbb{S}}$, $n \geqslant 1$. For any subdiagram $B \subseteq D$, there is a diagrammatic subalgebra $\overline{\mathfrak{g}}_B \subseteq \overline{\mathfrak{g}}$ with negative Borel subalgebra $\overline{\mathfrak{b}}_B \subseteq \overline{\mathfrak{b}}$. The corresponding root subsystem defines a universal subalgebra $\mathfrak{U}^n_{\mathbb{S},B} \subseteq \mathfrak{U}^n_{\mathbb{S}}$.

The definition of a braided pre–Coxeter structure on Drinfeld–Yetter modules over $\{\overline{\mathfrak{b}}_B\}_{B \subseteq D}$ can be lifted to an algebraic datum on $\mathfrak{U}^{\bullet}_{\mathbb{S}}$, which we call a *universal* braided pre–Coxeter structure. This consists of a collection of associators $\Phi_B \in \mathfrak{U}^3_{\mathbb{S},B}$, $B \subseteq D$, twists $J^{\mathcal{F}}_{B'B} \in \mathfrak{U}^2_{\mathbb{S},B}$, $\mathcal{F} \in \mathsf{Mns}(B, B')$, and gauge transformations $\Upsilon^{\mathcal{G}\mathcal{F}} \in \mathfrak{U}^1_{\mathbb{S},B}$, $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$.

1.14. By relying on the computation of the Hochschild cohomology of $\mathfrak{U}^{\bullet}_{\mathbb{S}}$, in particular the vanishing of $H^1(\mathfrak{U}^{\bullet}_{\mathbb{S}})$, we prove the uniqueness of universal braided pre–Coxeter structures on $\mathfrak{U}^{\bullet}_{\mathbb{S}}$ with prescribed braiding. We also show that a universal

[3]The data labeling the restriction functors $F^{\mathcal{F}}_{B'B}$ and isomorphisms $\Upsilon^{\mathcal{G}\mathcal{F}}$ actually consists of a *maximal nested set* on B relative to B' (see Section 11 for the definition). The collection of such nested sets is a quotient of the set of set of maximal chains, and for simplicity we identify the two in the introduction.

[4]Recall that $B' \perp B$ if $B \cap B' = \emptyset$ and no vertex of B is connected with one of B'.

braided pre–Coxeter structure with diagrammatic categories $\{\mathsf{DY}^{\hbar,\mathsf{int},0}_{\overline{\mathfrak{b}}_B}\}_{B\subseteq D}$ extends in at most one way to a braided Coxeter one. This gives our main result.

Theorem. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two universal braided Coxeter structures with diagrammatic categories $\{\mathsf{DY}^{\hbar,\mathsf{int},0}_{\overline{\mathfrak{b}}_B}\}_{B\subseteq D}$. Then,*

(1) *$\mathcal{C}_1$ and $\mathcal{C}_2$ are twist equivalent.*
(2) *The twist relating them is unique up to a unique gauge transformation.*

The categories $\mathsf{DY}^{\hbar,\mathsf{int},0}_{\overline{\mathfrak{b}}_B}$ naturally contains a generalisation $\mathcal{O}^{\hbar,\mathsf{int}}_{\infty,\overline{\mathfrak{g}}_B}$ of category $\mathcal{O}$, where the weight spaces are allowed to be infinite–dimensional. The theorem above readily restricts to the diagrammatic categories $\{\mathcal{O}^{\hbar,\mathsf{int}}_{\infty,\overline{\mathfrak{g}}_B}\}_{B\subseteq D}$ and yields the following.

Corollary. *There is, up to a unique universal equivalence, a unique universal braided Coxeter structure with diagrammatic categories $\{\mathcal{O}^{\hbar,\mathsf{int}}_{\infty,\overline{\mathfrak{g}}_B}\}_{B\subseteq D}$*

1.15. **Outline of the paper.** In Section 2, we review the Etingof–Kazhdan quantisation of Lie bialgebras and its description in terms of the PROP $\underline{\mathsf{LBA}}$ of Lie bialgebras. In Section 3, we review the theory of Schur functors and their cohomology following [4] and [12] respectively. In Section 4, we describe the factorised structure of morphisms in $\underline{\mathsf{LBA}}$, and their relation to free Lie algebras. In Section 5, we introduce the PROP $\underline{\mathsf{DY}}^n$, the algebra $\mathfrak{U}^n_{\mathsf{DY}}$, and we study its properties and its Hochschild cohomology. In Section 6, we introduce the refined PROP $\underline{\mathsf{PLBA}}$, describing a split inclusion $\mathfrak{a}\subset\mathfrak{b}$ of Lie bialgebras, and the corresponding universal algebra $\mathfrak{U}^n_{\mathsf{PDY}}$, for which we prove a number of results analogous to those obtained for $\mathfrak{U}^n_{\mathsf{DY}}$. In Section 7 these are used to prove the uniqueness, up to a unique gauge transformation, of the relative quantisation functor constructed in [1]. Section 8 contains some background material on partial semigroups and Lie bialgebras graded over them. In Section 9, we study the further refined PROP $\underline{\mathsf{LBA}}_{\mathsf{S}}$, for a partial abelian semigroup S, and its universal algebra $\mathfrak{U}^n_{\mathsf{S}}$. In particular, we compute its Hochschild cohomology. In Section 10, we study the subalgebras of $\mathfrak{U}^n_{\mathsf{S}}$ defined by the saturated subsemigroups of S. Section 11 reviews the combinatorial definitions of diagrams and maximal nested sets. In Section 12, we define diagrammatic (partial) semigroups, and define for these an extension of $\underline{\mathsf{LBA}}_{\mathsf{S}}$ which allows in particular to simultaneously account for both the diagrammatic structure of the Borel subalgebra of a complex semisimple Lie algebra as well as its root space decomposition. In Section 13 we define a universal braided pre–Coxeter structure associated to a diagrammatic semigroup, and prove its rigidity. In Section 14, we review the definition of braided (pre-)Coxeter categories following [2]. We then show that the braided pre–Coxeter structures introduced in Section 13 give rise to braided (pre–)Coxeter category structures on Drinfeld–Yetter modules over Lie bialgebras graded by a diagrammatic semigroup. In the final Section 15, we use these results to prove the uniqueness of braided pre–Coxeter structures on the category of integrable, Drinfeld–Yetter modules over the Borel subalgebra of an arbitrary symmetrisable diagrammatic or extended Kac–Moody algebra $\overline{\mathfrak{g}}$ and on category $\mathcal{O}$–modules over $\overline{\mathfrak{g}}$.

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2. Universal quantisation of Lie bialgebras

In this section, we review the Etingof–Kazhdan quantisation of Lie bialgebras, and its description in terms of *product–permutation categories* (PROPs). For more details, we refer the reader to [14, 15].

2.1. **Drinfeld double.** A Lie bialgebra over a field k is a triple $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ where $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}})$ is a Lie algebra (*i.e.,* $[\cdot,\cdot]_{\mathfrak{b}} : \mathfrak{b}\otimes\mathfrak{b}\to\mathfrak{b}$ is antisymmetric and satisfies the Jacobi identity), $(\mathfrak{b},\delta_{\mathfrak{b}})$ is a Lie coalgebra (*i.e.,* $\delta_{\mathfrak{b}} : \mathfrak{b}\to\mathfrak{b}\otimes\mathfrak{b}$ is antisymmetric and satisfies the co-Jacobi identity), and $[\cdot,\cdot]_{\mathfrak{b}}$, $\delta_{\mathfrak{b}}$ satisfy the cocycle condition

$$\delta_{\mathfrak{b}}([x,y]_{\mathfrak{b}}) = [x\otimes 1 + 1\otimes x, \delta_{\mathfrak{b}}(y)] - [y\otimes 1 + 1\otimes y, \delta_{\mathfrak{b}}(x)] \tag{2.1}$$

The Drinfeld double $\mathfrak{g}_{\mathfrak{a}}$ of $\mathfrak{b}$ is the Lie algebra defined as follows. As a vector space, $\mathfrak{g}_{\mathfrak{b}} = \mathfrak{b}\oplus\mathfrak{b}^*$. The pairing $\langle\cdot,\cdot\rangle : \mathfrak{b}\otimes\mathfrak{b}^*\to\mathsf{k}$ extends uniquely to a symmetric, non–degenerate bilinear form on $\mathfrak{g}_{\mathfrak{b}}$, such that $\mathfrak{b},\mathfrak{b}^*$ are isotropic subspaces. The Lie bracket on $\mathfrak{g}_{\mathfrak{b}}$ is then defined as the unique bracket compatible with $\langle\cdot,\cdot\rangle$, *i.e.,* such that

$$\langle [x,y], z\rangle = \langle x, [y,z]\rangle$$

for all $x,y,z\in\mathfrak{g}_{\mathfrak{b}}$. It coincides with $[\cdot,\cdot]_{\mathfrak{b}}$ on $\mathfrak{b}$, and with the bracket induced by $\delta_{\mathfrak{b}}$ on $\mathfrak{b}^*$. The mixed bracket for $b\in\mathfrak{b},\phi\in\mathfrak{b}^*$ is then equal to

$$[b,\phi] = \mathsf{ad}^*(b)(\phi) - \mathsf{ad}^*(\phi)(b) = \mathsf{ad}^*(b)(\phi) + \phi\otimes\mathrm{id}_{\mathfrak{b}}\circ\delta(b)$$

where ad^* denotes the coadjoint action of $\mathfrak{b}$ on $\mathfrak{b}^*$ and of $\mathfrak{b}^*$ on $\mathfrak{b}$, respectively.

The Lie algebra $\mathfrak{g}_{\mathfrak{b}}$ is a (topological) quasitriangular Lie bialgebra, with cobracket $\delta = \delta_{\mathfrak{b}}\oplus(-\delta_{\mathfrak{b}^*})$, where $\delta_{\mathfrak{b}^*}$ is the (topological) cobracket on $\mathfrak{b}^*$ induced by $[\cdot,\cdot]_{\mathfrak{b}}$, and r–matrix $r\in\mathfrak{g}_{\mathfrak{b}}\hat{\otimes}\mathfrak{g}_{\mathfrak{b}}$ corresponding to the identity in $\mathrm{End}(\mathfrak{b})\simeq\mathfrak{b}\hat{\otimes}\mathfrak{b}^*\subset\mathfrak{g}_{\mathfrak{b}}\hat{\otimes}\mathfrak{g}_{\mathfrak{b}}$. Explicitly, if $\{b_i\}_{i\in I},\{b^i\}_{i\in I}$ are dual bases of $\mathfrak{b}$ and $\mathfrak{b}^*$ respectively, then $r = \sum_{i\in I} b_i\otimes b^i\in\mathfrak{b}\hat{\otimes}\mathfrak{b}^*$.

2.2. **Drinfeld–Yetter modules.** A triple (V,π,π^*) is a Drinfeld–Yetter module over a Lie bialgebra $(\mathfrak{b},[\cdot,\cdot]_{\mathfrak{b}},\delta_{\mathfrak{b}})$ if (V,π) is a $\mathfrak{b}$–module, that is the map $\pi:\mathfrak{b}\otimes V\to V$ satisfies

$$\pi\circ[\cdot,\cdot]_{\mathfrak{b}} = \pi\circ(\mathrm{id}\otimes\pi) - \pi\circ(\mathrm{id}\otimes\pi)\circ(21) \tag{2.2}$$

(V,π^*) is a $\mathfrak{b}$–comodule, that is the map $\pi^*: V\to\mathfrak{b}\otimes V$ satisfies

$$\delta\circ\pi^* = (21)\circ(\mathrm{id}\otimes\pi^*)\circ\pi^* - (\mathrm{id}\otimes\pi^*)\circ\pi^* \tag{2.3}$$

and the maps π,π^* satisfy the following compatibility condition in $\mathrm{End}(\mathfrak{b}\otimes V)$:

$$\pi^*\circ\pi - \mathrm{id}\otimes\pi\circ(12)\circ\mathrm{id}\otimes\pi^* = [\cdot,\cdot]_{\mathfrak{b}}\otimes\mathrm{id}\circ\mathrm{id}\otimes\pi^* - \mathrm{id}\otimes\pi\circ\delta_{\mathfrak{b}}\otimes\mathrm{id} \tag{2.4}$$

The category $\mathsf{DY}_{\mathfrak{b}}$ is a symmetric tensor category.

In terms of representations of the Drinfeld double, $\mathsf{DY}_{\mathfrak{b}}$ is equivalent to the category $\mathcal{E}_{\mathfrak{g}_{\mathfrak{b}}}$ of equicontinuous $\mathfrak{g}_{\mathfrak{b}}$–modules [14]. Roughly speaking, a $\mathfrak{g}_{\mathfrak{b}}$–module is equicontinuous if the action of $\mathfrak{b}^*$ is locally finite. In particular, there is a functor $\mathcal{E}_{\mathfrak{g}_\mathfrak{b}} \to \mathsf{DY}_\mathfrak{b}$ which assign to any $(V,\pi) \in \mathcal{E}_{\mathfrak{g}_\mathfrak{b}}$, the Drinfeld–Yetter $\mathfrak{b}$–module (V,π,π^*) where π is restricted to $\mathfrak{b} \subset \mathfrak{g}_\mathfrak{b}$, and the coaction π^* is given by

$$\pi^*(v) = \sum_i b_i \otimes b^i \cdot v \in \mathfrak{b} \otimes V \tag{2.5}$$

The equicontinuity condition ensures that the sum is finite and the coaction well–defined. Conversely, given a Drinfeld–Yetter $\mathfrak{b}$–module (V,π,π^*), the action of $\phi \in \mathfrak{b}^*$ on V is defined by the formula

$$\phi \cdot v = \phi \otimes \mathrm{id}_V \circ \pi^*(v) \tag{2.6}$$

The compatibility condition (2.4) guarantees that this lifts to an equicontinuous action of the Drinfeld double $\mathfrak{g}_\mathfrak{b}$. One can prove that this is an equivalence of symmetric tensor categories.

2.3. **Restricted Drinfeld double.** Let $\mathfrak{b} = \bigoplus_{n\in\mathbb{N}} \mathfrak{b}_n$ be an $\mathbb{N}$–graded Lie bialgebra with finite–dimensional homogeneous components. Its restricted dual $\mathfrak{b}^\star = \bigoplus_{n\in\mathbb{N}} \mathfrak{b}_n^*$ and its restricted Drinfeld double $\mathfrak{g}_\mathfrak{b}^{\mathrm{res}} = \mathfrak{b} \oplus \mathfrak{b}^\star$ are also Lie bialgebras with cobrackets $\delta_{\mathfrak{b}^\star} = [\,,\,]_\mathfrak{b}^t$ and $\delta_{\mathfrak{g}_\mathfrak{b}^{\mathrm{res}}} = \delta_\mathfrak{b} - \delta_{\mathfrak{b}^\star}$, respectively. Moreover, since $\mathfrak{b}^\star$ is dense in $\mathfrak{b}^*$, the Lie algebra $\mathfrak{g}_\mathfrak{b}^{\mathrm{res}}$ is dense in $\mathfrak{g}_\mathfrak{b}$. Therefore, any continuous action of $\mathfrak{g}_\mathfrak{b}^{\mathrm{res}}$ extends automatically to one of $\mathfrak{g}_\mathfrak{b}$. One can show easily that this induces a canonical isomorphism $\mathcal{E}_{\mathfrak{g}_\mathfrak{b}} \simeq \mathcal{E}_{\mathfrak{g}_\mathfrak{b}^{\mathrm{res}}}$. In particular, one has $\mathsf{DY}_\mathfrak{b} \simeq \mathcal{E}_{\mathfrak{g}_\mathfrak{b}^{\mathrm{res}}}$.

2.4. **Completions.** Let $\mathsf{f}_\mathfrak{b} : \mathsf{DY}_\mathfrak{b} \to \mathrm{Vect}$, $\mathsf{f}_{\mathfrak{g}_\mathfrak{b}} : \mathcal{E}_{\mathfrak{g}_\mathfrak{b}} \to \mathrm{Vect}$ be the forgetful functors and $\mathcal{U}_\mathfrak{b} = \mathsf{End}\,(\mathsf{f}_\mathfrak{b})$, $\widehat{U\mathfrak{g}_\mathfrak{b}} = \mathsf{End}\,(\mathsf{f}_{\mathfrak{g}_\mathfrak{b}})$ the corresponding algebras of endomorphisms. Since the equivalence $\mathcal{E}_{\mathfrak{g}_\mathfrak{b}} \simeq \mathsf{DY}_\mathfrak{b}$ preserves the underlying vector space and commutes with the forgetful functors, there is a canonical isomorphism $\mathcal{U}_\mathfrak{b} \simeq \widehat{U\mathfrak{g}_\mathfrak{b}}$. In particular, we can think of $U\mathfrak{g}_\mathfrak{b}$ as a subalgebra in $\mathcal{U}_\mathfrak{b}$.

Since the equivalence preserves the tensor structure, the same identification holds for the n–folds forgetful functor $\mathsf{f}^{\boxtimes n}(V_1,\dots,V_n) = V_1 \otimes \cdots \otimes V_n$, *i.e.,*

$$\mathcal{U}_\mathfrak{b}^n = \mathsf{End}\left(\mathsf{f}_\mathfrak{b}^{\boxtimes n}\right) \simeq \mathsf{End}\left(\mathsf{f}_{\mathfrak{g}_\mathfrak{b}}^{\boxtimes n}\right) = \widehat{U\mathfrak{g}_\mathfrak{b}^{\otimes n}} \tag{2.7}$$

and we can consider $U\mathfrak{g}_\mathfrak{b}^{\otimes n}$ as a subalgebra in $\mathcal{U}_\mathfrak{b}^n$.

Under the identification (2.7), the r–matrix of $\mathfrak{g}_\mathfrak{b}$, $r_\mathfrak{b} = \sum_i b_i \otimes b^i \in \mathfrak{b}\widehat{\otimes}\mathfrak{b}^* \subset \mathfrak{g}_\mathfrak{b}\widehat{\otimes}\mathfrak{g}_\mathfrak{b}$, where $\{b_i\}$ and $\{b^i\}$ are dual bases of $\mathfrak{b}$ and $\mathfrak{b}^*$, corresponds to the element of $\mathcal{U}_\mathfrak{b}^2$ given by the maps $r_{VW} \in \mathrm{End}_\mathsf{k}(V \otimes W)$, $V, W \in \mathsf{DY}_\mathfrak{b}$, defined by

$$r_{VW} = \pi_V \otimes \mathrm{id} \circ (12) \circ \mathrm{id} \otimes \pi_W^* \tag{2.8}$$

2.5. **Etingof–Kazhdan quantisation.** In [14], Etingof and Kazhdan give an explicit procedure to construct a quantisation of $\mathfrak{b}$, that is a Hopf algebra $U_\hbar\mathfrak{b}$ over $\mathsf{k}[\![\hbar]\!]$ endowed with an isomorphism

$$U_\hbar\mathfrak{b}/\hbar U_\hbar\mathfrak{b} \simeq U\mathfrak{b}$$

of Hopf algebras, which induces the cobracket $\delta_\mathfrak{b}$ on $\mathfrak{b}$.

The construction proceeds as follows. One considers the Drinfeld category $\mathsf{DY}_\mathfrak{b}^\Phi$ of deformation Drinfeld–Yetter $\mathfrak{b}$–modules,*i.e.,* topologically free $\mathsf{k}[\![\hbar]\!]$–modules with a Drinfeld–Yetter structure over $\mathfrak{b}$, with associativity and commutativity constraints given by

$$\Phi_{UVW} = \Phi(\hbar\Omega_{12}, \hbar\Omega_{23}) \qquad \text{and} \qquad \beta_{VW} = (12) \circ e^{\hbar\Omega/2}$$

where $U, V, W \in \mathsf{DY}_{\mathfrak{b}}$, $\Omega = r + r^{21}$, and Φ is a fixed Lie associator. Let $\mathsf{f} : \mathsf{DY}_{\mathfrak{b}}^{\Phi} \to \mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$ be the forgetful functor. Etingof and Kazhdan construct an explicit tensor structure on f, *i.e.,* a collection of natural isomorphisms

$$J^{\text{EK}}_{VW} : \mathsf{f}(V) \otimes \mathsf{f}(W) \to \mathsf{f}(V \otimes W)$$

which are the identity modulo $\hbar$ and satisfy the relation

$$\mathsf{f}(\Phi) \circ J^{\text{EK}}_{U \otimes V, W} \circ (J^{\text{EK}}_{U,V} \otimes \text{id}) = J^{\text{EK}}_{U, V \otimes W} \circ (\text{id} \otimes J^{\text{EK}}_{V,W}) \tag{2.9}$$

in $\text{Hom}(\mathsf{f}(U) \otimes \mathsf{f}(V) \otimes \mathsf{f}(W), \mathsf{f}(U \otimes (V \otimes W)))$.

The algebra $\widehat{\mathcal{U}}_{\mathfrak{b}} = \mathsf{End}\,(\mathsf{f})$ is a topological Hopf algebra, with coproduct induced by the tensor product in $\mathsf{DY}_{\mathfrak{b}}$. Twisting $\widehat{\mathcal{U}}_{\mathfrak{b}}$ by J^{EK} produces a new Hopf algebra, with a coassociative deformation coproduct Δ_J. In order to produce a quantisation of $\mathfrak{b}$, one considers the Drinfeld–Yetter module corresponding to the Verma module

$$M_{\mathfrak{b}} = \text{Ind}_{\mathfrak{b}^*}^{\mathfrak{g}_{\mathfrak{b}}} \mathbb{C} \simeq U\mathfrak{b}$$

and shows that there is a natural embedding $\mathsf{f}(M_{\mathfrak{b}}) \subset \mathsf{End}\,(\mathsf{f})$. The coproduct Δ_J induces a coproduct on $\mathsf{f}(M_{\mathfrak{b}})$ which can explicitely computed as the composition

$$\mathsf{f}(M_{\mathfrak{b}}) \xrightarrow{\mathsf{f}(\Delta_0)} \mathsf{f}(M_{\mathfrak{b}} \otimes M_{\mathfrak{b}}) \xrightarrow{(J^{\text{EK}}_{M_{\mathfrak{b}}, M_{\mathfrak{b}}})^{-1}} \mathsf{f}(M_{\mathfrak{b}}) \otimes \mathsf{f}(M_{\mathfrak{b}})$$

This induces a Hopf algebra structure on the vector space $\mathsf{f}(M_{\mathfrak{b}}) \simeq U\mathfrak{b}[\![\hbar]\!]$, which quantizes the Lie bialgebra $\mathfrak{b}$. In [15], Etingof and Kazhdan showed that the construction of the quantum enveloping algebra $\mathsf{f}(M_{\mathfrak{b}})$ is universal. In 2.6–2.10, we explain the precise meaning of this statement.

2.6. PROPs [20, 21, 13, 1]**.** A PROP is a k–linear, strict, symmetric monoidal category P whose objects are the non–negative integers, and such that $[n] \otimes [m] = [n+m]$. In particular $[0]$ is the unit object, and $[1]^{\otimes n} = [n]$. A morphism of PROPs is a symmetric monoidal functor $\mathcal{G} : \mathsf{P} \to \mathsf{Q}$ which is the identity on objects, and is endowed with the trivial tensor structure

$$\text{id} : \mathcal{G}[m]_{\mathcal{C}} \otimes \mathcal{G}[n]_{\mathcal{C}} = [m]_{\mathcal{D}} \otimes [n]_{\mathcal{D}} = [m+n]_{\mathcal{D}} = \mathcal{G}([m+n]_{\mathcal{C}})$$

Fix henceforth a complete bracketing b_n on n letters for any $n \geqslant 2$, and set $\mathbf{b} = \{b_n\}_{n \geqslant 2}$. A *module* over P in a symmetric monoidal category $\mathcal{N}$ is a symmetric monoidal functor $(\mathcal{G}, J) : \mathsf{P} \to \mathcal{N}$ such that[5]

$$\mathcal{G}([n]) = \mathcal{G}([1])^{\otimes n}_{b_n}$$

and the following diagram is commutative

$$\begin{tikzcd}
\mathcal{G}([m]) \otimes \mathcal{G}([n]) \arrow[r, "J_{[m],[n]}"] \arrow[d, equal] & \mathcal{G}([m+n]) \arrow[d, equal] \\
\mathcal{G}([1])^{\otimes m}_{b_m} \otimes \mathcal{G}([1])^{\otimes n}_{b_n} \arrow[r, "\Phi"'] & \mathcal{G}([1])^{\otimes (m+n)}_{b_{m+n}}
\end{tikzcd}$$

where Φ is the associativity constraint in $\mathcal{N}$. A *morphism* of modules over P is a natural transformation of functors. The category of P–modules is denoted by $\mathsf{Fun}^{\otimes}_{\mathbf{b}}(\mathsf{P}, \mathcal{N})$.

[5]In a monoidal category $(\mathcal{C}, \otimes)$, $V^{\otimes n}_{b_n}$ denotes the n–fold tensor product of $V \in \mathcal{C}$ bracketed according to b_n. For example $V^{\otimes 3}_{(\bullet\bullet)\bullet} = (V \otimes V) \otimes V$.

2.7. **The Karoubi envelope.** Recall that the Karoubi envelope of a category $\mathcal{C}$ is the category $\mathsf{Kar}(\mathcal{C})$ whose objects are pairs (X,π), where $X\in\mathcal{C}$ and $\pi:X\to X$ is an idempotent. The morphisms in $\mathsf{Kar}(\mathcal{C})$ are defined as

$$\mathsf{Kar}(\mathcal{C})((X,\pi),(Y,\rho))=\{f\in\mathcal{C}(X,Y)\mid \rho\circ f=f=f\circ\pi\}$$

with $\mathrm{id}_{(X,\pi)}=\pi$. In particular, $\mathsf{Kar}(\mathcal{C})((X,\mathrm{id}),(Y,\mathrm{id}))=\mathcal{C}(X,Y)$, and the functor $\mathcal{C}\to\mathsf{Kar}(\mathcal{C})$, mapping $X\mapsto(X,\mathrm{id})$, $f\mapsto f$, is fully faithful.

Every idempotent in $\mathsf{Kar}(\mathcal{C})$ splits canonically. Namely, if $q\in\mathsf{Kar}(\mathcal{C})((X,\pi),(X,\pi))$ satisfies $q^2=q$, the maps

$$i=q:(X,q)\to(X,\pi)\qquad\text{and}\qquad p=q:(X,\pi)\to(X,q)$$

satisfy $i\circ p=q$ and $p\circ i=\mathrm{id}_{(X,q)}$.

We denote by $\underline{\mathsf{P}}$ the closure under infinite direct sums of the Karoubi completion of P. It is then clear that, if $\mathcal{N}$ is Karoubi complete, there is an essentially unique equivalence $\mathsf{Fun}^{\otimes}_{\mathbf{b}}(\underline{\mathsf{P}},\mathcal{N})\simeq\mathsf{Fun}^{\otimes}_{\mathbf{b}}(\mathsf{P},\mathcal{N})$.

2.8. **Example.** Let LA be the PROP generated by a morphism $\mu:[2]\to[1]$ subject to the relations

$$\mu\circ(\mathrm{id}_{[2]}+(1\,2))=0\qquad\text{and}\qquad\mu\circ(\mu\otimes\mathrm{id}_{[1]})\circ(\mathrm{id}_{[3]}+(1\,2\,3)+(3\,1\,2))=0 \tag{2.10}$$

as morphisms $[2]\to[1]$ and $[3]\to[1]$ respectively. Then, there is a canonical isomorphism of categories $\mathsf{Fun}_{\mathbf{b}}(\mathsf{LA},\mathrm{Vect}_{\mathsf{k}})\simeq\mathsf{LA}(\mathsf{k})$, where $\mathsf{LA}(\mathsf{k})$ is the category of Lie algebras over k.

2.9. **The PROPs $\underline{\mathsf{LCA}}$ and $\underline{\mathsf{LBA}}$.** The PROP of Lie coalgebras $\underline{\mathsf{LCA}}$ is generated by a morphism $\delta:[1]\to[2]$ satisfying

$$(\mathrm{id}_{[2]}+(1\,2))\circ\delta=0\qquad\text{and}\qquad(\mathrm{id}_{[3]}+(123)+(312))\circ(\delta\otimes\mathrm{id}_{[1]})\circ\delta=0 \tag{2.11}$$

There is a natural identification of PROPs

$$\Theta:\mathsf{LCA}\to\mathsf{LA}^{\mathrm{op}} \tag{2.12}$$

defined by $\Theta(\delta)=\mu$. The relation between the functor Θ and the standard duality between Lie algebras and Lie coalgebras is easily described. Let $\mathfrak{b}$ be a Lie algebra and $\mathfrak{c}$ a Lie coalgebra, with a compatible pairing $\langle,\rangle:\mathfrak{b}\otimes\mathfrak{c}\to\mathsf{k}$, *i.e.,* such that $\langle[b_1,b_2]_{\mathfrak{b}},c\rangle=\langle b_1\otimes b_2,\delta_{\mathfrak{c}}(c)\rangle$ for any $b_1,b_2\in\mathfrak{b}$ and $c\in\mathfrak{c}$. Let $\mathcal{G}_{\mathfrak{b}}:\underline{\mathsf{LA}}\to\mathrm{Vect}_{\mathsf{k}},\mathcal{G}_{\mathfrak{c}}:\underline{\mathsf{LCA}}\to\mathrm{Vect}_{\mathsf{k}}$ be the corresponding realisation functors, then for any $T\in\underline{\mathsf{LA}}([N],[n])$, $\underline{b}_N\coloneqq b_1\otimes\ldots\otimes b_N\in\mathfrak{b}^{\otimes N}$ and $\underline{c}_n\coloneqq c_1\otimes\ldots\otimes c_n\in\mathfrak{c}^{\otimes n}$, one has

$$\langle\mathcal{G}_{\mathfrak{b}}(T)(\underline{b}_N),\underline{c}_n\rangle=\langle\underline{b}_N,\mathcal{G}_{\mathfrak{c}}(\Theta(T))(\underline{c}_n)\rangle \tag{2.13}$$

Finally, the PROP of Lie bialgebras $\underline{\mathsf{LBA}}$ is generated by $\mu:[2]\to[1]$ and $\delta:[1]\to[2]$ satisfying (2.10), (2.11), and the *cocycle condition*

$$\delta\circ\mu=(\mathrm{id}_{[2]}-(21))\circ\mathrm{id}\otimes\mu\circ\delta\otimes\mathrm{id}\circ(\mathrm{id}_{[2]}-(21)) \tag{2.14}$$

2.10. **Etingof–Kazhdan quantisation in $\underline{\mathsf{LBA}}$.** In [15], Etingof and Kazhdan showed that the construction of $J=J^{\mathrm{EK}}_{M_{\mathfrak{b}},M_{\mathfrak{b}}}$ is universal, *i.e.,* that it can be realised in the PROP $\underline{\mathsf{LBA}}$. To this end, one first replaces the module $M_{\mathfrak{b}}$ with a universal Drinfeld–Yetter module in $\underline{\mathsf{LBA}}$, by constructing an action and a coaction of the Lie bialgebra $[1]\in\underline{\mathsf{LBA}}$ on $M\coloneqq S[1]$. The twist J is then defined, using the same formulae as in [14], as an element

$$J\in\underline{\mathsf{LBA}}(M\otimes M,M\otimes M)[\![\hbar]\!]$$

It induces a universal quantisation functor, that is a functor Q from the PROP of Hopf algebras[6] $\underline{\mathsf{HA}}$ to $\underline{\mathsf{LBA}}$, mapping $[1]_{\mathsf{HA}}$ to $S[1]_{\mathsf{LBA}}$.

A universal interpretation of the fiber functor $(\mathsf{f}, J^{\mathrm{EK}})$, rather than of the Hopf algebra $(\mathsf{f}(M_{\mathfrak{a}}), J^{-1}\mathsf{f}(\Delta_0))$ alone, will be given in the Section 5, by using the PROP of universal Drinfeld–Yetter modules.

3. Schur functors

We review in this section some basic facts about the cohomology of Schur functors which are due to Enriquez [12, Sec. 1], and will be used repeatedly. The exposition follows the approach to the theory of Schur functors of Baez and Trimble [4].

3.1. **Schur functors.** Let Cat be the 2–category of categories and SymCat the 2–category of k–linear, additive, Karoubi closed, symmetric monoidal categories.

Definition. A *Schur functor* is an endomorphism of the forgetful 2–functor $\mathsf{f} : \mathsf{SymCat} \to \mathsf{Cat}$. That is, a collection of endofunctors $F_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$ in Cat, indexed by objects in SymCat, and invertible natural transformations $F_{\mathcal{G}}$ in Cat,

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "\mathcal{G}"] \arrow[d, "F_{\mathsf{C}_1}"'] \arrow[dr, phantom, "\swarrow F_{\mathcal{G}}"] & \mathsf{C}_2 \arrow[d, "F_{\mathsf{C}_2}"] \\ \mathsf{C}_1 \arrow[r, "\mathcal{G}"'] & \mathsf{C}_2 \end{tikzcd} \tag{3.1}$$

indexed by functors $\mathcal{G} \in \mathsf{SymCat}(\mathsf{C}_1, \mathsf{C}_2)$, and such that $F_{\mathrm{id}_{\mathsf{C}}} = \mathrm{id}_{F_{\mathsf{C}}}$ and $F_{\mathcal{G}_2 \circ \mathcal{G}_1} = F_{\mathcal{G}_1} \circ_{\mathsf{h}} F_{\mathcal{G}_2}$, *i.e.*,

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "\mathcal{G}_1"] \arrow[d, "F_{\mathsf{C}_1}"'] \arrow[dr, phantom, "\swarrow F_{\mathcal{G}_1}"] & \mathsf{C}_2 \arrow[r, "\mathcal{G}_2"] \arrow[d, "F_{\mathsf{C}_2}"] \arrow[dr, phantom, "\swarrow F_{\mathcal{G}_2}"] & \mathsf{C}_3 \arrow[d, "F_{\mathsf{C}_3}"] \\ \mathsf{C}_1 \arrow[r, "\mathcal{G}_1"'] & \mathsf{C}_2 \arrow[r, "\mathcal{G}_2"'] & \mathsf{C}_3 \end{tikzcd} = \begin{tikzcd} \mathsf{C}_1 \arrow[r, "\mathcal{G}_2 \circ \mathcal{G}_1"] \arrow[d, "F_{\mathsf{C}_1}"'] \arrow[dr, phantom, "\swarrow F_{\mathcal{G}_2 \circ \mathcal{G}_1}"] & \mathsf{C}_3 \arrow[d, "F_{\mathsf{C}_3}"] \\ \mathsf{C}_1 \arrow[r, "\mathcal{G}_2 \circ \mathcal{G}_1"'] & \mathsf{C}_3 \end{tikzcd} \tag{3.2}$$

A morphism of Schur functors $\phi : F^1 \to F^2$ is a collection of natural transformation $\phi_{\mathsf{C}} : F^1_{\mathsf{C}} \to F^2_{\mathsf{C}}$, indexed by $\mathsf{C} \in \mathsf{SymCat}$, such that, for any functor $\mathcal{G} \in \mathsf{SymCat}(\mathsf{C}_1, \mathsf{C}_2)$,

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, equal] \arrow[d, "F^2_{\mathsf{C}_1}"'] \arrow[dr, phantom, "\swarrow \phi_{\mathsf{C}_1}"] & \mathsf{C}_1 \arrow[r, "\mathcal{G}"] \arrow[d, "F^1_{\mathsf{C}_1}"] \arrow[dr, phantom, "\swarrow F^1_{\mathcal{G}}"] & \mathsf{C}_2 \arrow[d, "F^1_{\mathsf{C}_2}"] \\ \mathsf{C}_1 \arrow[r, equal] & \mathsf{C}_1 \arrow[r, "\mathcal{G}"'] & \mathsf{C}_2 \end{tikzcd} = \begin{tikzcd} \mathsf{C}_1 \arrow[r, "\mathcal{G}"] \arrow[d, "F^2_{\mathsf{C}_1}"'] \arrow[dr, phantom, "\swarrow F^2_{\mathcal{G}}"] & \mathsf{C}_2 \arrow[r, equal] \arrow[d, "F^2_{\mathsf{C}_1}"] \arrow[dr, phantom, "\swarrow \phi_{\mathsf{C}_2}"] & \mathsf{C}_2 \arrow[d, "F^1_{\mathsf{C}_2}"] \\ \mathsf{C}_1 \arrow[r, "\mathcal{G}"'] & \mathsf{C}_2 \arrow[r, equal] & \mathsf{C}_2 \end{tikzcd} \tag{3.3}$$

The category of Schur functors $\mathsf{Sch} = \mathsf{End}(\mathsf{f})$ is endowed with the following operations:

- **Direct sum.** For any $F_1, F_2 \in \mathsf{Sch}$, we set
$$F^1 \oplus F^2 = \oplus \circ F^1 \times F^2 \circ \triangle \tag{3.4}$$
where $\oplus : \mathsf{f} \times \mathsf{f} \to \mathsf{f}$ is thought of as a morphism of 2–functors, and $\triangle : \mathsf{f} \to \mathsf{f} \times \mathsf{f}$ is the diagonal. The neutral element is the zero functor $\Sigma_0 \in \mathsf{Sch}$, which assigns to each object in C the zero object in C.

[6] The PROP $\underline{\mathsf{HA}}$ is generated by the morphisms $m : [2] \to [1]$, $\iota : [0] \to [1]$, $\Delta : [2] \to [1]$, $\epsilon : [1] \to [0]$, $S, S^{-1} : [1] \to [1]$ with the relations coming from the Hopf algebra axioms.

- **Tensor product.** For any $F_1, F_2 \in \mathsf{Sch}$, we set
$$F^1 \otimes F^2 = \otimes \circ F^1 \times F^2 \circ \triangle \tag{3.5}$$
where, as before, $\otimes : \mathsf{f} \times \mathsf{f} \to \mathsf{f}$. The neutral element is the unit functor $T^0 \in \mathsf{Sch}$, which assigns to each object in C the unit object in C.

Both assignments extend to natural transformations and give rise to functors $\oplus, \otimes : \mathsf{Sch} \times \mathsf{Sch} \to \mathsf{Sch}$, which endow Sch with a natural structure of additive symmetric monoidal category.

3.2. **Representability.** Let $\mathsf{k}\mathfrak{S}$ denotes the permutation algebroid (*i.e.,* the free PROP generated by permutations) and $\overline{\mathsf{k}\mathfrak{S}}$ be its additive and Karoubian envelope.

Theorem. [4] *The forgetful* 2*–functor* $\mathsf{f} : \mathsf{SymCat} \to \mathsf{Cat}$ *is represented by* $\overline{\mathsf{k}\mathfrak{S}}$*, i.e., there is an equivalence of* 2*–functors*
$$\mathsf{f} \simeq \mathsf{SymCat}(\overline{\mathsf{k}\mathfrak{S}}, -) \tag{3.6}$$
In particular, $\mathsf{Sch} \simeq \overline{\mathsf{k}\mathfrak{S}}$ *in* Cat.

Proof. The proof is essentially a 2–categorical version of the representability of the forgetful functor from the category of representations of an associative algebra to vector spaces. Namely, for any $\mathsf{C} \in \mathsf{SymCat}$, there is a canonical functor $\mathsf{U}_\mathsf{C} : \mathsf{SymCat}(\overline{\mathsf{k}\mathfrak{S}}, \mathsf{C}) \to \mathsf{C}$ defined by $\mathsf{U}_\mathsf{C}(\mathcal{G}) = \mathcal{G}[1]$, for any $\mathcal{G} \in \mathsf{SymCat}(\overline{\mathsf{k}\mathfrak{S}}, \mathsf{C})$, and $\mathsf{U}_\mathsf{C}(\phi) = \phi_{[1]} : \mathcal{G}[1] \to \mathcal{G}'[1]$, for any natural transformation $\phi : \mathcal{G} \Rightarrow \mathcal{G}'$.

For any functor $\mathcal{F} : \mathsf{C}_1 \to \mathsf{C}_2$, the natural transformation $\mathsf{U}_\mathcal{F}$
$$\begin{tikzcd} \mathsf{SymCat}(\overline{\mathsf{k}\mathfrak{S}}, \mathsf{C}_1) \arrow[r, "\mathsf{U}_{\mathsf{C}_1}"] \arrow[d, "\mathcal{F}_*"'] & \mathsf{C}_1 \arrow[d, "\mathcal{F}"] \arrow[dl, Rightarrow, "\mathsf{U}_\mathcal{F}"'] \\ \mathsf{SymCat}(\overline{\mathsf{k}\mathfrak{S}}, \mathsf{C}_2) \arrow[r, "\mathsf{U}_{\mathsf{C}_2}"] & \mathsf{C}_2 \end{tikzcd} \tag{3.7}$$
is given by the identity on $\mathcal{F} \circ \mathcal{G}[1]$.

It is easy to see that this defines an essentially unique equivalence of 2–functors $\mathsf{f} \simeq \mathsf{SymCat}(\overline{\mathsf{k}\mathfrak{S}}, -)$. It then follows from Yoneda lemma $\mathsf{Sch} = \mathsf{End}\,(\mathsf{f}) \simeq \overline{\mathsf{k}\mathfrak{S}}$. □

3.3. **Abelianity.** We will use of the following general fact.

Proposition. *Let* A *be an associative* k*–algebra,* C_A *the corresponding algebroid (i.e.,* C_A *is the category with one object* $\bullet$ *and* $\mathsf{End}_{\mathsf{C}_A}(\bullet) = A$*), and* $\overline{\mathsf{C}}_A$ *its additive and Karoubi envelope. Then* $\overline{\mathsf{C}}_A$ *is equivalent to the category* $\mathsf{Proj}(A^{\mathrm{op}})$ *of projective* A^{op}*–modules.*

Proof. Let $\mathsf{C}_A^\oplus$ be the additive envelope of C_A. Then the functor $\mathsf{C}_A^\oplus \to \mathrm{Rep}\, A^{\mathrm{op}}$ mapping the generating object [1] to A induces an equivalence of categories $\mathsf{C}_A^\oplus \simeq \mathsf{Free}(A^{\mathrm{op}})$. It follows
$$\overline{\mathsf{C}}_A = \mathsf{Kar}(\mathsf{C}_A^\oplus) \simeq \mathsf{Kar}(\mathsf{Free}(A^{\mathrm{op}})) \simeq \mathsf{Proj}(A^{\mathrm{op}}) \tag{3.8}$$
□

Corollary. *If* A *is hereditary (resp. semisimple),* $\overline{\mathsf{C}}_A$ *is abelian (resp. semisimple). In particular, the category* $\overline{\mathsf{k}\mathfrak{S}}$*, and therefore* Sch*, is a semisimple category.*

3.4. **Representations of $\mathfrak{S}_n$.** Recall that the set of irreducible representations $\widehat{\mathfrak{S}}_n$ of the symmetric group $\mathfrak{S}_n$ is in bijection with minimal idempotents in $k\mathfrak{S}_n$, modulo the equivalence relation $p \sim upu^{-1}$, $u \in k\mathfrak{S}_n^\times$. We henceforth regard $\widehat{\mathfrak{S}}_n$ as a subset of $k\mathfrak{S}_n$ by choosing a representative for each class, and set $\widehat{\mathfrak{S}} = \bigsqcup_{n\geqslant 0} \widehat{\mathfrak{S}}_n$. If $\pi \in \widehat{\mathfrak{S}}_n$, we set $|\pi| = n$. We proved in 3.3 that the category of Schur functor is semisimple and equivalent to category $\mathrm{Rep}\,\mathsf{k}\mathfrak{S}$ of representations of $\mathsf{k}\mathfrak{S} = \bigoplus_N \mathsf{k}\mathfrak{S}_N$. It follows that, up to isomorphism, any Schur functor has the form:

$$F_{\mathsf{C}}(X) = \bigoplus_{\pi\in\widehat{\mathfrak{S}}} \pi\left(X^{\otimes|\pi|}\right)^{\oplus m_\pi} \tag{3.9}$$

for some $m_\pi \in \mathbb{N} \cup \{\infty\}$.

3.5. **Schur bifunctors.** A *Schur bifunctor* is a morphism of 2–functors from $\mathsf{f} \times \mathsf{f}$ to f, where $\mathsf{f} \times \mathsf{f}(\mathsf{C}) = \mathsf{C} \times \mathsf{C}$. The category of Schur bifunctors is denoted $\mathsf{Sch}_2 = \mathsf{Hom}(\mathsf{f} \times \mathsf{f}, \mathsf{f})$. In particular, $\oplus$ and $\otimes$ are Schur bifunctors.

Schur bifunctors can be obtained from Schur functors by using the following operations.

- **External tensor product.** For any $F^1, F^2 \in \mathsf{Sch}$, set
$$F^1 \boxtimes F^2 = \otimes \circ F^1 \times F^2$$
- **Coproduct.** For any $F \in \mathsf{Sch}$, set
$$\Delta(F) = F \circ \oplus$$

Example. If $S = \bigoplus_{n\geqslant 0} S^n$ and $\wedge = \bigoplus_{n\geqslant 0} \wedge^n$ are the symmetric and exterior algebra functors, then

$$\Delta(S) \simeq S \boxtimes S \qquad \text{and} \qquad \Delta(\wedge) \simeq \wedge \boxtimes \wedge$$

The results from 3.2 and 3.3 readily extends to Sch_2.

Theorem.

(1) *The 2–functor $\mathsf{f}\times\mathsf{f} : \mathsf{SymCat} \to \mathsf{Cat}$ is represented by the category $\overline{\mathsf{k}\mathfrak{S}} \times \overline{\mathsf{k}\mathfrak{S}}$, i.e., there is an equivalence*
$$\mathsf{f} \times \mathsf{f} \simeq \mathsf{SymCat}(\overline{\mathsf{k}\mathfrak{S}} \times \overline{\mathsf{k}\mathfrak{S}}, -) \tag{3.10}$$
(2) $\mathsf{Sch}_2 \simeq \overline{\mathsf{k}\mathfrak{S}} \times \overline{\mathsf{k}\mathfrak{S}}$ *in* Cat.
(3) Sch_2 *is a semisimple abelian category.*

Proof. The representability of $\mathsf{f} \times \mathsf{f}$ is straightforward. Then, by Yoneda lemma, one gets

$$\mathsf{Sch}_2 = \mathsf{Hom}(\mathsf{f} \times \mathsf{f}, \mathsf{f}) \simeq \overline{\mathsf{k}\mathfrak{S}} \times \overline{\mathsf{k}\mathfrak{S}}$$

From Corollary 3.3, we conclude that Sch_2 is semisimple and abelian. □

3.6. **Cohomology of Schur (bi)functors.** Since the category of Schur (bi)functors is abelian, we can consider the cohomology of complexes in Sch or Sch_2.

Proposition. [12, Prop. 1.3] *Let $(F^n, d^n)_{n\geqslant 0}$ be a complex in* Sch. *Then*

$$H^i(\Delta(F^\bullet), \Delta(d^\bullet)) \simeq \Delta(H^i(F^\bullet, d^\bullet))$$

Proof. It is enough to observe that the functor $\Delta = -\circ\oplus : \mathsf{Sch} \to \mathsf{Sch}_2$ is additive, and therefore exact, due to the fact that Sch and Sch_2 are semisimple. □

3.7. **The Hochschild complex.** The Hochschild complex $(SV^{\otimes\bullet}, d_H)$ of a symmetric coalgebra SV can be interpreted as a complex of Schur functors as follows.

Let Σ_1 be the Schur functor $\mathrm{id}_{\mathsf{f}} \in \mathsf{Sch}$ and, for any $n \geqslant 1$, set $\Sigma_n = \Sigma_1^{\oplus n} = \Sigma_1 \oplus \cdots \oplus \Sigma_1$. Let $i_0 : \Sigma_0 \to \Sigma_1$ be the inclusion of the zero object $\Sigma_0 \in \mathsf{Sch}$, and $\delta : \Sigma_1 \to \Sigma_2$ the diagonal morphism, *i.e.,* for any $\mathsf{C} \in \mathsf{SymCat}$ and $X \in \mathsf{C}$, $\delta_{\mathsf{C},X} = (\mathrm{id}_X, \mathrm{id}_X) : X \to X \oplus X$. There are natural transformations $\{\delta_n^i\}_{i=0}^{n+1} : \Sigma_n \to \Sigma_{n+1}$ defined as follows [7]

$$(i=0) \qquad \Sigma_n = \Sigma_0 \oplus \Sigma_n \xrightarrow{i_0 \oplus \mathrm{id}_{\Sigma_n}} \Sigma_1 \oplus \Sigma_n = \Sigma_{n+1}$$

$$(i=n+1) \qquad \Sigma_n = \Sigma_n \oplus \Sigma_0 \xrightarrow{\mathrm{id}_{\Sigma_n} \oplus i_0} \Sigma_n \oplus \Sigma_1 = \Sigma_{n+1}$$

and, for $1 \leqslant i \leqslant n$,

$$\Sigma_n = \Sigma_{i-1} \oplus \Sigma_1 \oplus \Sigma_{n-i} \xrightarrow{\mathrm{id}_{\Sigma_{i-1}} \oplus \delta \oplus \mathrm{id}_{\Sigma_{n-i}}} \Sigma_{i-1} \oplus \Sigma_2 \oplus \Sigma_{n-i} = \Sigma_{n+1}$$

The natural transformations $\{\delta_i^n\}$ give rise to a cosimplicial structure on the tower of Schur functors $S^{\otimes n} = S \circ \Sigma_n$, whose associated differential is the Hochschild differential d_H. The latter restricts to zero on $T^\bullet \subset S^{\otimes\bullet}$, where $T^n = \Sigma_1 \otimes \cdots \otimes \Sigma_1$, and gives rise to a quasi–isomorphism

$$\iota : (\wedge^\bullet, 0) \to (S^{\otimes\bullet}, d_H)$$

3.8. **The diagonal Hochschild complex.** By Proposition 3.6, the map $\Delta(\iota)$ induces a quasi–isomorphism[8]

$$H^n(S^{\otimes\bullet} \boxtimes S^{\otimes\bullet}, d_H \boxtimes d_H) = H^n(\Delta(S^{\otimes\bullet}), \Delta(d_H)) \simeq \Delta(\wedge^n) = \bigoplus_{j=0}^{n} \wedge^j \boxtimes \wedge^{n-j} \tag{3.11}$$

To work out $\Delta(\imath)$ explicitly, note that it arises from the restriction to $\Delta(\wedge^n)$ of the inclusion $d_n = \Delta(\iota_1^{\otimes n}) : \Delta(T^n) \subset \Delta(S^{\otimes n}) \cong S^{\otimes n} \boxtimes S^{\otimes n}$, where $\iota_1 : T \to S$ is the inclusion. The restriction of d_n to $T^j \boxtimes T^{n-j} \subset \Delta(T^n)$ is readily seen to be

$$\tau_j = ({\iota_1}^{\otimes j} \otimes {\iota_0}^{\otimes n-j}) \boxtimes ({\iota_0}^{\otimes j} \otimes \iota_1^{\otimes n-j})$$

where $\iota_0 : T^0 \to S$ is the inclusion of the unit. Since d_n is equivariant under $\mathfrak{S}_n$, its restriction to $\wedge^j \boxtimes \wedge^{n-j} \subset \Delta(\wedge^n) \subset \Delta(T^n)$ is given by

$$\mathrm{Alt}_n^{(2)} \circ \tau_j \circ \mathrm{Alt}_j \otimes \mathrm{Alt}_{n-j} \tag{3.12}$$

where $\mathrm{Alt}_n = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma \sigma$ and $\mathrm{Alt}_n^{(2)} = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma \sigma \boxtimes \sigma$.

[7] In the category Vect, the Schur functor $\Sigma_n : \mathrm{Vect} \to \mathrm{Vect}$ is given by $V \to V^{\oplus n} = V \otimes \mathsf{k}^n$, and the natural transformations $\{\delta_i^n\}_{i=0}^{n+1} : \Sigma_n \to \Sigma_{n+1}$ are induced by the maps $\mathsf{k}^n \to \mathsf{k}^{n+1}$ given by

$$(x_1, \ldots, x_n) \to \left\{ \begin{array}{cc} (0, x_1, \ldots, x_n) & i=0 \\ (x_1, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n) & 1 \leqslant i \leqslant n \\ (x_1, \ldots, x_n, 0) & i = n+1 \end{array} \right.$$

[8] Here, and in the sequel, the notation $S^{\otimes\bullet} \boxtimes S^{\otimes\bullet}$ refers to the complex whose nth term is $S^{\otimes n} \boxtimes S^{\otimes n}$, not to the total complex underlying the exterior product of the complex $S^{\otimes\bullet}$ with itself.

Example. For $n=2$, the restriction of $\Delta(\iota)$ to $\wedge^1 \boxtimes \wedge^1$ is given by

$$\mathrm{Alt}_2^{(2)} \circ \tau_1 = \frac{1}{2}\left[(\iota_1 \otimes \iota_0) \boxtimes (\iota_0 \otimes \iota_1) - (\iota_0 \otimes \iota_1) \boxtimes (\iota_1 \otimes \iota_0)\right]$$

In Vect, this reads: for any $V, W \in \mathrm{Vect}$, $\Delta(\iota)$ is given on $V \otimes W \subset \wedge^2(V \oplus W)$ by

$$v \otimes w \mapsto \frac{1}{2}\left[(v \otimes 1) \otimes (1 \otimes w) - (1 \otimes v) \otimes (w \otimes 1)\right]$$

3.9. **Tensor algebra.** Set $T^0 = 1$, *i.e.*, the unit object in Sch, and $T^n = \Sigma_1^{\otimes n} = \Sigma_1 \otimes \cdots \otimes \Sigma_1$ for any $n \geqslant 1$. By Theorem 3.2, $\mathsf{Sch}(T^n, T^n) = \mathsf{k}\mathfrak{S}_n$. The tensor algebra functor $T = \bigoplus_{n \geqslant 0} T^n$ is also endowed with a cosimplicial structure on $\{T^{\otimes \bullet}\}$. Namely, let $\Delta^{\mathsf{sh}} : T \to T \otimes T$ be the *shuffle coproduct* defined on T^n by

$$\Delta^{\mathsf{sh}}_n = \sum_{\substack{n_1+n_2=n \\ \sigma \in \mathsf{Sh}(n_1,n_2)}} \psi_{n_1,n_2} \circ \sigma : T^n \to \bigoplus_{n_1+n_2=n} T^{n_1} \otimes T^{n_2} \subset T \otimes T \tag{3.13}$$

where $\mathsf{Sh}(n_1,n_2) \subset \mathfrak{S}_n$ is the set of (n_1,n_2)–shuffles, *i.e.*, permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma(i) < \sigma(j)$ whenever $1 \leqslant i < j \leqslant n_1$ or $n_1 + 1 \leqslant i < j \leqslant n$, and ψ_{n_1,n_2} is the *deconcatenation* $T^n = T^{n_1} \otimes T^{n_2}$. Then the Hochschild differential on T is $d_H^n = \sum_{i=0}^{n+1} d_i^n$, with face maps $\{d_i^n\}_{i=0}^{n+1} : T^{\otimes n} \to T^{\otimes(n+1)}$ given by

$$d_i^n = \begin{cases} 1 \otimes \mathrm{id}_{T^{\otimes n}} & i = 0 \\ \mathrm{id}_{T^{\otimes(i-1)}} \otimes \Delta^{\mathsf{sh}} \otimes \mathrm{id}_{T^{\otimes(n-i)}} & i = 1, \dots, n-1 \\ \mathrm{id}_{T^{\otimes n}} \otimes 1 & i = n \end{cases}$$

The canonical inclusion $\mathrm{Sym} : S \hookrightarrow T$ preserves the differential defined on 3.7. and induces a morphism of complexes $(S^{\otimes \bullet}, d_H) \to (T^{\otimes \bullet}, d_H)$.

3.10. **Duality in Sch.** Let Sch^{Π} be the completion of Sch under infinite direct products. That is, we formally add to Sch the objects $\prod_i F_i \in \mathsf{Sch}^{\mathbb{N}}$ with morphisms

$$\mathsf{Sch}^{\Pi}\left(\prod_i F_i, \prod_j F'_j\right) \quad = \quad \prod_j \bigoplus_i \mathsf{Sch}^{\Pi}\left(F_i, F'_j\right) \tag{3.14}$$

Objects in Sch^{Π} are well–defined, and universal, on symmetric monoidal categories closed under infinite direct products. The equivalence $\mathsf{Sch} \simeq \mathrm{Rep}\,\mathsf{k}\mathfrak{S}$ extends the duality in $\mathrm{Rep}\,\mathsf{k}\mathfrak{S}$ to a contravariant functor $\mathsf{Sch} \to \mathsf{Sch}^{\Pi}$, where

$$\left(\bigoplus_i F_i\right)^* = \prod_i F_i^*$$

3.11. PROP**s and Schur bifunctors.** A PROP P gives rise to a functor $\mathsf{P}_{\mathsf{Sch}} : \mathsf{Sch}_2 \to \mathrm{Vect}$ which is defined on a bifuntor $F = \bigoplus_i F_i \boxtimes G_i$ by [9]

$$\mathsf{P}_{\mathsf{Sch}}(F) = \bigoplus_i \underline{\mathsf{P}}(F_i^*[1], G_i[1]) \tag{3.15}$$

Then for any morphism

$$f = \sum_{i,j} f_{ij} \boxtimes g_{ij} : F = \bigoplus_i F_i \boxtimes G_i \to F' = \bigoplus_j F'_j \boxtimes G'_j$$

[9] The definition of $\mathsf{P}_{\mathsf{Sch}}$ requires to consider the completion of P with respect to infinite direct limits, which, by abuse of notation, we still denote by $\underline{\mathsf{P}}$.

we define $\mathsf{P}_{\mathsf{Sch}}(f):\mathsf{P}_{\mathsf{Sch}}(F)\to\mathsf{P}_{\mathsf{Sch}}(F')$ as follows. For any $\phi=\sum_i\phi_i$ in $\mathsf{P}_{\mathsf{Sch}}(F)=\bigoplus_i\underline{\mathsf{P}}(F_i^*[1],G_i[1])$, we set

$$\mathsf{P}_{\mathsf{Sch}}(f)(\phi)=\sum_j\left(\sum_i g_{ij}\circ\phi_i\circ f_{ij}^*\right)\in\bigoplus_j\mathsf{P}((F_j')^*[1],G_j[1])=\mathsf{P}_{\mathsf{Sch}}(F')$$

Proposition. [12, Prop. 1.2] *For any complex* $(F^\bullet,d^\bullet)$ *in* Sch_2, $(\mathsf{P}_{\mathsf{Sch}}(F^\bullet),\mathsf{P}_{\mathsf{Sch}}(d^\bullet))$ *is a complex of vector spaces, and*

$$H^i(\mathsf{P}_{\mathsf{Sch}}(F^\bullet),\mathsf{P}_{\mathsf{Sch}}(d^\bullet))\simeq\mathsf{P}_{\mathsf{Sch}}(H^i(F^\bullet,d^\bullet))$$

Proof. It is enough to observe that the functor $\mathsf{P}_{\mathsf{Sch}}:\mathsf{Sch}_2\to\mathrm{Vect}$ is additive, and therefore exact. □

3.12. **Hochschild cohomology.** The differential $\mathsf{P}_{\mathsf{Sch}}(d_H\boxtimes d_H)$ of the complex $\mathsf{P}_{\mathsf{Sch}}(S^{\otimes\bullet}\boxtimes S^{\otimes\bullet})$ can be described more explicitely. From 3.10, we have

$$S^*=\prod_{n\geqslant 0}S^n=:\widehat{S}\tag{3.16}$$

and the cosimplicial structure on $S^{\otimes\bullet}$ described in 3.7 induces a simplicial structure on $\widehat{S}^{\otimes\bullet}$ with associated differential ∂_H. Therefore, for any $\phi\in\mathsf{P}_{\mathsf{Sch}}(\widehat{S}^{\otimes n}[1],S^{\otimes n}[1])$, one has

$$\mathsf{P}_{\mathsf{Sch}}(d_H\boxtimes d_H)(\phi)=d_H\circ\phi\circ\partial_H$$

Analogous considerations hold for the complex of Schur bifunctors $(T^{\otimes\bullet}\boxtimes T^{\otimes\bullet},d_H\boxtimes d_H)$.

Proposition. *The following holds for any* PROP P.

(1) *The inclusion*

$$\underline{\mathsf{P}}(\widehat{S}[1]^{\otimes\bullet},S[1]^{\otimes\bullet})\to\underline{\mathsf{P}}(\widehat{T}[1]^{\otimes\bullet},T[1]^{\otimes\bullet})$$

induced by the natural inclusion $\mathrm{Sym}:S\to T$ *and projection* $\mathrm{Sym}:\widehat{T}\to\widehat{S}$, *is a morphism of cosimplicial spaces.*

(2) *The inclusion*

$$\mathsf{P}_{\mathsf{Sch}}(\widetilde{\imath}):\left(\bigoplus_{j=0}^{\bullet}\underline{\mathsf{P}}\left(\wedge^j[1],\wedge^{\bullet-j}[1]\right),0\right)\to\left(\underline{\mathsf{P}}(\widehat{S}[1]^{\otimes\bullet},S[1]^{\otimes\bullet}),d_H\circ(-)\circ\partial_H\right)$$

obtained by (3.12) *is a quasi–isomorphism.*

Proof. (1) It is enough to observe that, by duality, $\mathrm{Sym}:\widehat{T}\to\widehat{S}$ induces a morphism of simplicial objects.

(2) We have

$$\begin{aligned}H^i\left(\underline{\mathsf{P}}(\widehat{S}[1]^{\otimes\bullet},S[1]^{\otimes\bullet}),d_H\circ(-)\circ\partial_H\right)&=H^i\left(\mathsf{P}_{\mathsf{Sch}}(S^{\otimes\bullet}\boxtimes S^{\otimes\bullet}),\mathsf{P}_{\mathsf{Sch}}(d_H\boxtimes d_H)\right)\\&=\mathsf{P}_{\mathsf{Sch}}\left(H^i\left(S^{\otimes\bullet}\boxtimes S^{\otimes\bullet},d_H\boxtimes d_H\right)\right)\\&=\mathsf{P}_{\mathsf{Sch}}\left(\bigoplus_{j=0}^{i}\wedge^j\boxtimes\wedge^{i-j}\right)\\&=\bigoplus_{j=0}^{i}\underline{\mathsf{P}}\left(\wedge^j[1],\wedge^{i-j}[1]\right)\end{aligned}$$

where the first and last equalities hold by definition of the functor $\mathsf{P}_{\mathsf{Sch}}$, the second one by Proposition 3.11, and the third one by (3.11). $\square$

The quasi–isomorphism $\mathsf{P}_{\mathsf{Sch}}(\widetilde{\iota})$ is described as follows. Let $\iota_0 : T^0 \to S$, $\iota_1 : T^1 \to S$ be the canonical inclusions, and $\iota_0^* : \widehat{S} \to T^0$, $\iota_1^* : \widehat{S} \to T^1$ the corresponding projections. Set $\tau_j' : T^j \to S^{\otimes i}$ by $\tau_j' = \iota_1^{\otimes j} \otimes \iota_0^{\otimes i-j}$, $\tau_{i-j}'' : \widehat{S} \to T^{i-j}$ by $\tau_{i-j}'' = \iota_0^{\otimes j} \otimes \iota_1^{\otimes i-j}$, and let $\widetilde{\tau}_j' : \wedge^j \to S^{\otimes i}$, $\widetilde{\tau}_{i-j}'' : \widehat{S}^{\otimes i} \to \wedge^{n-j}$ be the compositions with Alt_j and Alt_{n-j}, respectively. Then, for any $\phi \in \underline{\mathsf{P}}(\wedge^j[1], \wedge^{i-j}[1])$, $\mathsf{P}_{\mathsf{Sch}}(\widetilde{\iota})(\phi) \in \underline{\mathsf{P}}(\widehat{S}[1]^{\otimes i}, S[1]^{\otimes i})$ is given by

$$\mathsf{P}_{\mathsf{Sch}}(\phi) = \frac{1}{i!} \sum_{\sigma \in \mathfrak{S}_i} (-1)^{\sigma} \sigma \circ \widetilde{\tau}_{n-j}'' \circ \phi \circ \widetilde{\tau}_j' \circ \sigma^{-1} \tag{3.17}$$

4. Factorisation of morphisms in $\underline{\mathsf{LBA}}$

In 4.1–4.4, we review the polarised structure of morphisms in the PROP $\underline{\mathsf{LBA}}$, and their relation to free Lie algebras obtained in [10, 22]. We include proofs for the reader's convenience, and because they readily carry over to the refinements of $\underline{\mathsf{LBA}}$ introduced in Sections 6 and 9.

4.1. **Factorisation of morphisms in $\underline{\mathsf{LBA}}$.** The inclusions $\underline{\mathsf{LCA}}, \underline{\mathsf{LA}} \subset \underline{\mathsf{LBA}}$ induce maps

$$i_{p,q}^N : \underline{\mathsf{LCA}}([p],[N]) \otimes \underline{\mathsf{LA}}([N],[q]) \to \underline{\mathsf{LBA}}([p],[N]) \otimes \underline{\mathsf{LBA}}([N],[q]) \to \underline{\mathsf{LBA}}([p],[q])$$

given by the composition of morphisms in $\underline{\mathsf{LBA}}$.

Proposition. *The maps $\{i_{p,q}^N\}_{N\geqslant 0}$ induce an isomorphism*

$$\underline{\mathsf{LBA}}([p],[q]) \simeq \bigoplus_{N\geqslant 0} \underline{\mathsf{LCA}}([p],[N]) \otimes_{\mathfrak{S}_N} \underline{\mathsf{LA}}([N],[q])$$

Proof. Morphisms in $\underline{\mathsf{LBA}}$ can be represented as linear combinations of oriented graphs with no loops or multiple edges, obtained by (horizontal) composition

$$\underline{\mathsf{LBA}}([p],[q]) \otimes \underline{\mathsf{LBA}}([q],[s]) \stackrel{\circ^{op}}{\to} \underline{\mathsf{LBA}}([p],[s])$$

or tensor product (vertical composition)

$$\underline{\mathsf{LBA}}([p],[q]) \otimes \underline{\mathsf{LBA}}([p'],[q']) \stackrel{\otimes}{\to} \underline{\mathsf{LBA}}([p+p'],[q+q'])$$

The cocycle condition (2.14) allows to reorder every morphism as a linear combination of diagrams where the cobrakets horizontally precede the brackets. Finally, all permutations can be moved after the cobrackets and before the brackets, and identified with elements in $\mathfrak{S}_N$.

The decomposition in terms of the morphisms in the PROP $\underline{\mathsf{LA}}$ and $\underline{\mathsf{LCA}}$ follows, and the tensor product in the proposition should be interpreted as horizontal composition of graphs. The natural map to $\underline{\mathsf{LBA}}$ factors through the simultaneous action of $\mathfrak{S}_N$, and provides a surjective map.

The injectivity follows by the evaluation of the morphism in $\underline{\mathsf{LBA}}$ on the Lie bialgebra $F(\mathfrak{c}) = T\mathfrak{c}$, obtained from a Lie coalgebra $(\mathfrak{c}, \delta)$ with the free Lie algebra structure. $\square$

4.2. **Morphisms in $\underline{\mathsf{LA}}$ and $\underline{\mathsf{LCA}}$.** Let $\mathcal{L}_N$ be the free Lie algebra over k with generators $x_1, \dots, x_N$. The relation between $\mathcal{L}_N$ and morphisms in the PROPs $\underline{\mathsf{LA}}, \underline{\mathsf{LCA}}$ is easily explained by considering the following description of $\mathcal{L}_N$ in terms of binary trees (see, *e.g.*, [23, §0.2]).

Let $\mathsf{T}(N)$ denote the set of binary trees over X, recursively defined as follows: $x_1, \dots, x_N \in \mathsf{T}(N)$ and, for any $t_1, t_2 \in \mathsf{T}(N)$, $(t_1, t_2) \in \mathsf{T}(N)$. Let $\mathcal{T}_N$ denote the k–vector space with basis $\mathsf{T}(N)$. The composition law $(\cdot,\cdot)$ extends to a bilinear mapping $(\cdot,\cdot) : \mathcal{T}_N \otimes \mathcal{T}_N \to \mathcal{T}_N$. Let $J \subset \mathcal{T}_N$ be the ideal generated by all elements of the form (t,t), $t \in \mathcal{T}_N$, and $(t_1,(t_2,t_3))+(t_2,(t_3,t_1))+(t_3,(t_1,t_2))$, $t_1,t_2,t_3 \in \mathcal{T}_N$, and set $\mathcal{L}_N = \mathcal{T}_N/J$. It is easy to see that $\mathcal{L}_N$ is the free Lie algebra over X. We consider on $\mathcal{L}_N$ the natural $\mathbb{N}^N$–grading given by $\deg(x_i) = e_i$.

Lemma. *There are natural isomorphisms, compatible with the actions of $\mathfrak{S}_N$ and $\mathfrak{S}_n$*

$$\underline{\mathsf{LA}}([N],[n]) \simeq (\mathcal{L}_N^{\otimes n})_{\delta_N} \simeq \underline{\mathsf{LCA}}([n],[N])$$

where $\delta_N = e_1 + \cdots + e_N$, and $(\mathcal{L}_N^{\otimes n})_{\delta_N}$ is the subspace of its n–fold tensor product spanned by homogeneous elements of degree one in each variable.

Proof. The identification with $\underline{\mathsf{LA}}([N],[n])$ is straightforward. We first observe that

$$\underline{\mathsf{LA}}([N],[n]) \simeq \bigoplus_{(I_1,\dots,I_n)\in\mathsf{P}(\mathsf{N},\mathsf{n})} \underline{\mathsf{LA}}([|I_1|],[1]) \otimes \cdots \otimes \underline{\mathsf{LA}}([|I_n|],[1]) \tag{4.1}$$

where $\mathsf{P}(\mathsf{N},\mathsf{n})$ is the set of partitions of $\{1,\dots,N\}$ by n unordered sets.

Assume now that $n = 1$. Every morphism in $\underline{\mathsf{LA}}([N],[1])$ is represented by a linear combination of trees with N leaves precomposed with a permutation $\sigma \in \mathfrak{S}_N$. The permutation σ determines uniquely a labeling by $\{x_1,\dots,x_N\}$, where the ith leaf is labeled by $x_{\sigma^{-1}(i)}$. This provides a surjective map from $(\mathcal{L}_N)_{\delta_N}$ to $\underline{\mathsf{LA}}([N],[1])$. Conversely, every morphism $f \in \underline{\mathsf{LA}}([N],[1])$ determines an element in $(\mathcal{L}_N)_{\delta_N}$ by evaluating f on $x_1 \otimes \cdots \otimes x_N \in \mathcal{L}_N^{\otimes N}$, and the two maps are inverses of each other. Combined with (4.1), this extends to a canonical isomorphism $\underline{\mathsf{LA}}([N],[n]) \simeq (\mathcal{L}_N^{\otimes n})_{\delta_N}$. The identification with $\underline{\mathsf{LCA}}([n],[N])$ follows by the equivalence $\mathsf{LCA} \simeq \mathsf{LA}^{\mathrm{op}}$. □

4.3. **Morphisms in $\underline{\mathsf{LBA}}$ and free Lie algebras.**

Proposition.

(1) *There is an isomorphism of $(\mathfrak{S}_q, \mathfrak{S}_p)$–bimodules*

$$\underline{\mathsf{LBA}}([p],[q]) \simeq \bigoplus_{N\geqslant 1} \left((\mathcal{L}_N^{\otimes p})_{\delta_N} \otimes (\mathcal{L}_N^{\otimes q})_{\delta_N}\right)_{\mathfrak{S}_N}$$

(2) *Let $F \in k\mathfrak{S}_p$ and $G \in k\mathfrak{S}_q$ be idempotents, and $F[p] = ([p], F)$, $G[q] = ([q], G)$ the corresponding objects in $\underline{\mathsf{LBA}}$. Then one has*

$$\underline{\mathsf{LBA}}(F[p], G[q]) \simeq \bigoplus_{N\geqslant 0} \left(F(\mathcal{L}_N^{\otimes p})_{\delta_N} \otimes G(\mathcal{L}_N^{\otimes q})_{\delta_N}\right)_{\mathfrak{S}_N}$$

Proof. (1) follows from Proposition 4.1 and Lemma 4.2. (2) Normal ordering in $\underline{\mathsf{LBA}}$ gives

$$\underline{\mathsf{LBA}}(F[p], G[q]) \simeq \bigoplus_{N \geqslant 0} \underline{\mathsf{LCA}}(F[p], [N]) \otimes_{\mathfrak{S}_N} \underline{\mathsf{LA}}([N], G[q])$$

By 4.2 and 2.7, $\underline{\mathsf{LCA}}(F[p], [N]) \simeq F(\mathcal{L}_N^{\otimes p})_{\delta_N}$ and $\underline{\mathsf{LA}}([N], G[q]) \simeq G(\mathcal{L}_N^{\otimes q})_{\delta_N}$. □

4.4. **The tensor and symmetric algebras in $\underline{\mathsf{LBA}}$.** The objects $T[1], S[1]$ in $\underline{\mathsf{LBA}}$ play an important role in understanding the structure of the universal algebras which will be introduced in Section 5,

Let $A = \bigoplus_{p \geqslant 0} A^p \in \underline{\mathsf{LBA}}$ be either $T[1]$ or $S[1]$. If follows from 3.12 that the tower $\underline{\mathsf{LBA}}(\widehat{A}^{\otimes n}, A^{\otimes n})$ has a cosimplicial structure, and the map

$$\mathrm{Sym} : \underline{\mathsf{LBA}}(\widehat{S}[1]^{\otimes n}, S[1]^{\otimes n}) \to \underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \tag{4.2}$$

obtained by combining the natural projection $\widehat{T}[1] \to \widehat{S}[1]$ and injection $S[1] \to T[1]$ is a morphism of cosimplicial spaces.

The following result relates this structure to the standard cosimplicial structure on the tensor and symmetric algebras of the free Lie algebras $\mathcal{L}_N$ via the identifications provided by Proposition 4.3.

Lemma. *Let* $\mathsf{Sym} : S\mathcal{L}_N \to T\mathcal{L}_N$ *be the symmetrisation map. The following is a commutative diagram of cosimplicial spaces*

$$\begin{tikzcd}
\underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \arrow[r] & \bigoplus_{N \geqslant 0} \left((T\mathcal{L}_N^{\otimes n})_{\delta_N} \otimes (T\mathcal{L}_N^{\otimes n})_{\delta_N} \right)_{\mathfrak{S}_N} \\
\underline{\mathsf{LBA}}(\widehat{S}[1]^{\otimes n}, S[1]^{\otimes n}) \arrow[r] \arrow[u, "\mathsf{Sym}"] & \bigoplus_{N \geqslant 0} \left((S\mathcal{L}_N^{\otimes n})_{\delta_N} \otimes (S\mathcal{L}_N^{\otimes n})_{\delta_N} \right)_{\mathfrak{S}_N} \arrow[u, "\mathsf{Sym}\otimes\mathsf{Sym}"']
\end{tikzcd}$$

where the horizontal maps are those defined in Proposition 4.3.

5. Universal Drinfeld–Yetter modules

We introduce in this section the PROP $\underline{\mathsf{DY}}^n$ describing n Drinfeld–Yetter modules $[\mathsf{V}_1], \ldots, [\mathsf{V}_n]$ over a Lie bialgebra. The algebra

$$\mathfrak{U}^n_{\mathsf{DY}} = \mathrm{End}_{\underline{\mathsf{DY}}^n}([\mathsf{V}_1] \otimes \cdots \otimes [\mathsf{V}_n])$$

is universal in that, for any Lie bialgebra $\mathfrak{b}$ with Drinfeld double $\mathfrak{g}_{\mathfrak{b}}$, it is endowed with a canonical morphism $\mathfrak{U}^n_{\mathsf{DY}} \longrightarrow \widehat{U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}}$ to the completion of the n–fold tensor product of the enveloping algebra of $\mathfrak{g}_{\mathfrak{b}}$ considered in 2.4. We show that the tower $\{\mathfrak{U}^n_{\mathsf{DY}}\}_{n \geqslant 0}$ shares many properties of $\{U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}\}_{n \geqslant 0}$, namely that it has a cosimplicial structure, satisfies the PBW theorem, and that its Hochschild cohomology is given by a universal version of the exterior algebra of $\mathfrak{g}_{\mathfrak{b}}$.

5.1. **Colored PROPs.** A *colored* PROP P is a k–linear, strict, symmetric monoidal category whose objects are finite sequences over a set A, *i.e.,*

$$\mathsf{Obj}(\mathsf{P}) = \coprod_{n \geq 0} \mathsf{A}^n$$

with tensor product given by concatenation of sequences, and tensor unit given by the empty sequence. Modules over a colored PROP are defined as in 2.6.

5.2. **The PROP $\underline{\mathsf{DY}}^n$ and the algebra $\mathfrak{U}^n_{\mathsf{DY}}$.**

Definition. Let $n \geq 1$.

(1) $\underline{\mathsf{DY}}^n$ is the colored PROP generated by $n+1$ objects $[1]$ and $\{[\mathsf{V}_k]\}_{k=1}^n$, and morphisms

$$\mu : [2] \to [1] \qquad \delta : [1] \to [2]$$
$$\pi_k : [1] \otimes [\mathsf{V}_k] \to [\mathsf{V}_k] \qquad \pi_k^* : [\mathsf{V}_k] \to [1] \otimes [\mathsf{V}_k]$$

such that $([1], \mu, \delta)$ is a Lie bialgebra in $\underline{\mathsf{DY}}^n$, and every $([\mathsf{V}_k], \pi_k, \pi_k^*)$ is a Drinfeld–Yetter module over $[1]$.

(2) $\mathfrak{U}^n_{\mathsf{DY}}$ is the algebra given by

$$\mathfrak{U}^n_{\mathsf{DY}} = \mathsf{End}_{\underline{\mathsf{DY}}^n}\left([\mathsf{V}_1] \otimes \cdots \otimes [\mathsf{V}_n]\right)$$

If $\mathcal{N}$ is a k–linear symmetric monoidal category, the category of $\underline{\mathsf{DY}}^n$–modules in $\mathcal{N}$ is isomorphic to the category whose objects are tuples $(\mathfrak{b}; V_1, \dots, V_n)$ consisting of a Lie bialgebra $\mathfrak{b}$ in $\mathcal{N}$, and n Drinfeld–Yetter modules $V_1, \dots, V_n \in \mathcal{N}$ over $\mathfrak{b}$. A morphism $(\mathfrak{b}; V_1, \dots, V_n) \mapsto (\mathfrak{c}; W_1, \dots, W_n)$ is a tuple $(\phi; f_1, \dots, f_n)$, where $\phi : \mathfrak{b} \to \mathfrak{c}$ is a morphism of Lie bialgebras, and $f_i : V_i \to W_i$ are such that the following diagrams are commutative

$$\begin{tikzcd}
\mathfrak{b} \otimes V_i \arrow[r, "\pi_{V_i}"] \arrow[d, "\phi\otimes f_i"'] & V_i \arrow[d, "f_i"] \\
\mathfrak{c} \otimes W_i \arrow[r, "\pi_{W_i}"'] & W_i
\end{tikzcd}
\qquad
\begin{tikzcd}
V_i \arrow[r, "\pi^*_{V_i}"] \arrow[d, "f_i"'] & \mathfrak{b} \otimes V_i \arrow[d, "\phi\otimes f_i"] \\
W_i \arrow[r, "\pi^*_{W_i}"'] & \mathfrak{c} \otimes W_i
\end{tikzcd}$$

so that f_i is a morphism of $\mathfrak{b}$–modules $V_i \to \phi^* W_i$ as well as a morphism of $\mathfrak{c}$–comodules $\phi_* V_i \to W_i$.

5.3. **Action of $\mathfrak{U}^n_{\mathsf{DY}}$ on Drinfeld–Yetter modules.** Let $(\mathfrak{b}, [\cdot,\cdot], \delta) \in \mathsf{Vect}_\mathsf{k}$ be a Lie bialgebra, $\{V_k, \pi_k, \pi_k^*\}_{k=1}^n$ n Drinfeld–Yetter modules over $\mathfrak{b}$, and

$$\mathcal{G}_{(\mathfrak{b}, V_1, \dots, V_n)} : \underline{\mathsf{DY}}^n \longrightarrow \mathsf{Vect}_\mathsf{k}$$

the corresponding (symmetric, monoidal) realisation functor such that $[1] \mapsto \mathfrak{b}$ and $[\mathsf{V}_k] \mapsto V_k$.

Proposition. *Let $\mathsf{f}\colon \mathsf{DY}_\mathfrak{b} \to \mathsf{Vect}_\mathsf{k}$ be the forgetful functor, and $\mathcal{U}^n_\mathfrak{b} := \mathsf{End}\left(\mathsf{f}^{\boxtimes n}\right)$. Then, there is an algebra homomorphism*

$$\rho^n_\mathfrak{b} : \mathfrak{U}^n_{\mathsf{DY}} \to \mathcal{U}^n_\mathfrak{b}$$

which assigns to any $T \in \mathfrak{U}^n_{\mathsf{DY}}$, and any $V_1, \dots, V_n \in \mathsf{DY}_\mathfrak{b}$ the endomorphism $\mathcal{G}_{(\mathfrak{b}, V_1, \dots, V_n)}(T) \in \mathrm{End}_\mathsf{k}(V_1 \otimes \cdots \otimes V_n)$.

Proof. We need to prove that, for any $\{V_i, W_i\}_{i=1}^n \subset \mathsf{DY}_\mathfrak{b}$ and $f_i \in \mathrm{Hom}_{\mathsf{DY}_\mathfrak{b}}(V_i, W_i)$, one has

$$f_1 \otimes \cdots \otimes f_n \circ \mathcal{G}_{(\mathfrak{b}, V_1, \dots, V_n)}(T) = \mathcal{G}_{(\mathfrak{b}, W_1, \dots, W_n)}(T) \circ f_1 \otimes \cdots \otimes f_n$$

This follows from the fact that $(\mathrm{id}_\mathfrak{b}; f_1, \dots, f_n)$ induces a natural transformation $\mathcal{G}_{(\mathfrak{b}, V_1, \dots, V_n)} \Rightarrow \mathcal{G}_{(\mathfrak{b}, W_1, \dots, W_n)}$. $\square$

5.4. **Distinguished elements in $\mathfrak{U}_{\mathsf{DY}}$.** The algebra $\mathfrak{U}^2_{\mathsf{DY}}$ has a distinguished element, which is given by

$$r_{[\mathsf{V}_1],[\mathsf{V}_2]} = \pi_{[\mathsf{V}_1]} \otimes \mathrm{id}_{[\mathsf{V}_2]} \circ (12) \circ \mathrm{id}_{[\mathsf{V}_1]} \otimes \pi^*_{[\mathsf{V}_2]}$$

and is easily seen to be a solution of the classical Yang–Baxter equation in $\mathfrak{U}^3_{\mathsf{DY}}$

$$[r_{[\mathsf{V}_1],[\mathsf{V}_2]}, r_{[\mathsf{V}_1],[\mathsf{V}_3]}] + [r_{[\mathsf{V}_1],[\mathsf{V}_2]}, r_{[\mathsf{V}_2],[\mathsf{V}_3]}] + [r_{[\mathsf{V}_1],[\mathsf{V}_3]}, r_{[\mathsf{V}_1],[\mathsf{V}_3]}] = 0$$

Under the homomorphism $\rho^2_{\mathfrak{b}} : \mathfrak{U}^2_{\mathsf{DY}} \to \mathcal{U}^2_{\mathfrak{b}}$, $r_{[\mathsf{V}_1],[\mathsf{V}_2]}$ corresponds to the action of the r–matrix $r_{\mathfrak{b}} = \sum_i b_i \otimes b^i$ of $\mathfrak{g}_{\mathfrak{b}}$ defined in (2.8).

The algebra $\mathfrak{U}_{\mathsf{DY}}$ contains the element $\kappa = \pi_{[\mathsf{V}_1]} \circ \pi^*_{[\mathsf{V}_1]}$, which corresponds to the normally ordered Casimir operator $\kappa_{\mathfrak{b}} = \sum_i b_i b^i = m(r_{\mathfrak{b}})$ of $\mathfrak{g}_{\mathfrak{b}}$. We note further that while one can consider the following elements in $\mathfrak{U}^2_{\mathsf{DY}}$

$$r_{21} = \mathrm{id}_{[\mathsf{V}_1]} \otimes \pi_{[\mathsf{V}_2]} \circ\ (12) \circ \pi^*_{[\mathsf{V}_1]} \otimes \mathrm{id}_{[V_2]}$$

$$\kappa_1 = \left(\pi_{[\mathsf{V}_1]} \circ \pi^*_{[\mathsf{V}_1]}\right) \otimes \mathrm{id}_{[\mathsf{V}_2]} \qquad \text{and} \qquad \kappa_2 = \mathrm{id}_{[\mathsf{V}_1]} \otimes \left(\pi_{[V_2]} \circ \pi^*_{[V_2]}\right)$$

which correspond to $r^{21}_{\mathfrak{b}}$, $\kappa_{\mathfrak{b}} \otimes 1$ and $1 \otimes \kappa_{\mathfrak{b}}$ in $\mathcal{U}^2_{\mathfrak{b}}$ respectively, there is no analogue in $\mathfrak{U}_{\mathsf{DY}}$ of the non–normally ordered Casimir operator $\sum_i b^i b_i = m(r^{21}_{\mathfrak{b}})$, which does not converge in $\mathcal{U}^2_{\mathfrak{b}}$ if $\dim \mathfrak{b} = +\infty$.

5.5. **Universal invariants.**

Definition. An element $\phi \in \mathfrak{U}^n_{\mathsf{DY}}$ is *invariant* if it commutes with the action and coaction of the Lie bialgebra [1] on $[\mathsf{V}_1] \otimes \cdots \otimes [\mathsf{V}_n]$, that is satisfies

$$\pi_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \mathrm{id}_{[1]} \otimes \phi = \phi \circ \pi_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]}$$

as maps $[1] \otimes [\mathsf{V}_1] \otimes \cdots \otimes [\mathsf{V}_n] \to [\mathsf{V}_1] \otimes \cdots \otimes [\mathsf{V}_n]$, and

$$\pi^*_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \phi = \mathrm{id}_{[1]} \otimes \phi \circ \pi^*_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]}$$

as maps $[\mathsf{V}_1] \otimes \cdots \otimes [\mathsf{V}_n] \to [1] \otimes [\mathsf{V}_1] \otimes \cdots \otimes [\mathsf{V}_n]$.

Let $(\mathfrak{U}^n_{\mathsf{DY}})^{\mathsf{inv}} \subset \mathfrak{U}^n_{\mathsf{DY}}$ be the subalgebra of invariant elements. The following is clear.

Proposition. *The map $\rho^n_{\mathfrak{b}} : \mathfrak{U}^n_{\mathsf{DY}} \to \mathcal{U}^n_{\mathfrak{b}}$ defined in 5.3 restricts to a homomorphism*

$$\rho^n_{\mathfrak{b}} : (\mathfrak{U}^n_{\mathsf{DY}})^{\mathsf{inv}} \to (\mathcal{U}^n_{\mathfrak{b}})^{\mathsf{inv}} := \mathsf{End}\left(\mathsf{id}^{\boxtimes \mathsf{n}}\right)$$

5.6. **Cosimplicial structure of $\mathcal{U}_{\mathfrak{b}}$.** The monoidal structure on $\mathsf{DY}_{\mathfrak{b}}$ endows the tower $\{\mathcal{U}^n_{\mathfrak{b}}\}$ with the structure of a cosimplicial complex of algebras

$$\mathsf{k} \rightrightarrows \mathcal{U}_{\mathfrak{b}} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathcal{U}^2_{\mathfrak{b}} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathcal{U}^3_{\mathfrak{b}} \quad \cdots$$

The corresponding face morphisms $\{d^n_i\}_{i=0}^{n+1} : \mathcal{U}^n_{\mathfrak{b}} \to \mathcal{U}^{n+1}_{\mathfrak{b}}$, are given by $(d^0_0\varphi)_V = (d^0_1\varphi)_V = \varphi \cdot \mathrm{id}_V$, for $\varphi \in \mathsf{k}$ and $V \in \mathsf{DY}_{\mathfrak{b}}$, and, for $n \geqslant 1$, $\varphi \in \mathcal{U}^n_{\mathfrak{b}}$, and $\{V_i\}_{i=1}^{n+1} \subset\in \mathsf{DY}_{\mathfrak{b}}$

$$(d^n_i\varphi)_{V_1,\dots,V_{n+1}} = \begin{cases} \mathrm{id}_{V_1} \otimes \varphi_{V_2,\dots,V_{n+1}} & i = 0 \\ \varphi_{V_1,\dots,V_i \otimes V_{i+1},\dots,V_{n+1}} & 1 \leqslant i \leqslant n \\ \varphi_{V_1,\dots,V_n} \otimes \mathrm{id}_{V_{n+1}} & i = n+1 \end{cases}$$

The degeneration homomorphisms $\varepsilon^i_n : \mathcal{U}^n_{\mathfrak{b}} \to \mathcal{U}^{n-1}_{\mathfrak{b}}$, for $i = 1, \dots, n$, are

$$(\varepsilon^i_n\varphi)_{X_1,\dots,X_{n-1}} = \varphi_{X_1,\dots,X_{i-1},\mathbf{1},X_i,\dots,X_{n-1}}$$

The morphisms $\{\varepsilon_n^i\}, \{d_i^n\}$ satisfy the standard relations

$$\begin{aligned} d_{n+1}^j d_n^i &= d_{n+1}^i d_n^{j-1} \quad & i < j \\ \varepsilon_n^j \varepsilon_{n+1}^i &= \varepsilon_n^i \varepsilon_{n+1}^{j+1} \quad & i \leqslant j \end{aligned}$$

and

$$\varepsilon_{n+1}^j d_i^n = \begin{cases} d_{n-1}^i \varepsilon_n^{j-1} & i < j \\ \mathrm{id} & i = j, j+1 \\ d_{n-1}^{i-1} \varepsilon_n^j & i > j+1 \end{cases}$$

These give rise to the Hochschild differential

$$d^n = \sum_{i=0}^{n+1} (-1)^i d_i^n : \mathcal{U}_{\mathfrak{b}}^n \to \mathcal{U}_{\mathfrak{b}}^{n+1}$$

5.7. **Cosimplicial structure of $\mathfrak{U}_{\mathsf{DY}}^n$.** The above structures can be lifted to the PROPs $\underline{\mathsf{DY}}^n$. For every $n \geqslant 1$ and $i = 0, 1, \dots, n+1$, there are faithful functors

$$\mathcal{D}_i^n : \underline{\mathsf{DY}}^n \to \underline{\mathsf{DY}}^{n+1}$$

mapping [1] to [1], and given by

$$\mathcal{D}_0^n([\mathsf{V}_k]) = [\mathsf{V}_{k+1}] \qquad \text{and} \qquad \mathcal{D}_{n+1}^n([\mathsf{V}_k]) = [\mathsf{V}_k]$$

for $1 \leqslant k \leqslant n$, and, for $1 \leqslant i \leqslant n$,

$$\mathcal{D}_i^n([\mathsf{V}_k]) = \begin{cases} [\mathsf{V}_k] & 1 \leqslant k \leqslant i-1 \\ [\mathsf{V}_i] \otimes [\mathsf{V}_{i+1}] & k = i \\ [\mathsf{V}_{k+1}] & i+1 \leqslant k \leqslant n \end{cases}$$

and $\mathcal{E}_n^i : \underline{\mathsf{DY}}^n \to \underline{\mathsf{DY}}^{n-1}$

$$\mathcal{E}_n^i = \mathcal{G}_{([1],[\mathsf{V}_1],\dots,[\mathsf{V}_{i-1}],\mathbf{1},[\mathsf{V}_{i+1}],\dots,[\mathsf{V}_{n-1}])}$$

where $\mathbf{1}$ is the trivial representation in $\underline{\mathsf{DY}}^n$. These induce algebra homomorphisms

$$\Delta_i^n : \mathfrak{U}_{\mathsf{DY}}^n \to \mathfrak{U}_{\mathsf{DY}}^{n+1}$$

which are universal analogues of the insertion/coproduct maps on $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}$. They give the tower of algebras $\{\mathfrak{U}_{\mathsf{DY}}^n\}_{n\geqslant 0}$ the structure of a cosimplicial complex, with Hochschild differential $d^n = \sum_{i=0}^{n+1}(-1)^i \Delta_i^n : \mathfrak{U}_{\mathsf{DY}}^n \to \mathfrak{U}_{\mathsf{DY}}^{n+1}$. The morphisms $\rho_{\mathfrak{b}}^n : \mathfrak{U}_{\mathsf{DY}}^n \to \mathcal{U}_{\mathfrak{b}}^n$ defined in 5.2 are compatible with the face and degeneration morphisms, and therefore with the differentials d^n.

5.8. **The algebra $T[1]$ and the coalgebra $\widehat{T}[1]$.** Regard $T[1] = \bigoplus_{p\geqslant 0}[p]$ as a graded algebra, with concatenation product given by the identification $[p_1] \otimes [p_2] = [p_1 + p_2]$ and unit given by the embedding $\imath : [0] \hookrightarrow T[1]$. If $([V], \pi_{[V]})$ is a module over the Lie algebra [1] in $\underline{\mathsf{DY}}^1$, the iterated action maps

$$\pi_{[V]}^{(p)} : [p] \otimes [V] \to [V]$$

endow $[V]$ with the structure of a module over $T[1]$.

Dually, regard $\widehat{T}[1] = \prod_{p\geqslant 0}[p]$ as a topological graded coalgebra, with deconcatenation coproduct given by the direct sum of identifications

$$[p] \to \bigoplus_{p_1+p_2=p} [p_1] \otimes [p_2] = \bigoplus_{p_1+p_2=p} [p]$$

and counit given by the projection $\epsilon : \widehat{T}[1] \to [0]$. Then, if $([V], \pi^*_{[V]})$ is a comodule over the Lie coalgebra $[1]$ in $\underline{\mathsf{DY}}^1$, the iterated coaction maps

$$\pi^{*(p)} : [V] \to [p] \otimes [V]$$

endow $[V]$ with the structure of a comodule over $\widehat{T}[1]$.

5.9. **Action of morphisms in $\underline{\mathsf{LBA}}$ on $\mathfrak{U}^n_{\mathsf{DY}}$.** Consider now the vector space

$$\underline{\mathsf{LBA}}(\widehat{T}[1], T[1]) = \bigoplus_{p,q\geqslant 0} \underline{\mathsf{LBA}}([p],[q])$$

with the convolution product $\phi_1 \star \phi_2 = m_{T[1]} \circ \phi_1 \otimes \phi_2 \circ \Delta_{\widehat{T}[1]}$, and unit $1_{T[1]} \circ \epsilon_{\widehat{T}[1]}$. Then, regarding $[\mathsf{V}_1] \in \underline{\mathsf{DY}}^1$ as a module over $T[1]$ and a comodule over $\widehat{T}[1]$ yields a convolution action of $\underline{\mathsf{LBA}}(\widehat{T}[1], T[1])$ on $\underline{\mathsf{DY}}^1([\mathsf{V}_1],[\mathsf{V}_1])$ given by

$$\phi \cdot X = \pi_{T[1]} \circ \phi \otimes X \circ \pi^*_{\widehat{T}[1]}$$

In particular, specialising to $X = \mathrm{id}_{[\mathsf{V}_1]}$ yields a map

$$\mathsf{a}^1 : \underline{\mathsf{LBA}}(\widehat{T}[1], T[1]) \longrightarrow \mathfrak{U}^1_{\mathsf{DY}} = \underline{\mathsf{DY}}^1([\mathsf{V}_1],[\mathsf{V}_1])$$

mapping ϕ to $\phi \cdot \mathrm{id}_{[\mathsf{V}_1]}$.

More generally, for any $n \geqslant 1$, the algebra structure on $T[1]^{\otimes n}$ and the coalgebra structure on $\widehat{T}[1]^{\otimes n}$ yield a map

$$\mathsf{a}^n : \underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \longrightarrow \mathfrak{U}^n_{\mathsf{DY}} = \underline{\mathsf{DY}}^n(\otimes_{k=1}^n [\mathsf{V}_k], \otimes_{k=1}^n [\mathsf{V}_k]),$$

which maps ϕ to $\phi \cdot \mathrm{id}_{\otimes_{k=1}^n [\mathsf{V}_k]}$.

Recall from 4.4 that the tower $\underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n})$ is cosimplicial.

Proposition. *The collection of maps $\{\mathsf{a}^n\}$ is a morphism of cosimplicial spaces.*

Proof. It suffices to prove that

$$\mathsf{a}^n : \underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \to \underline{\mathsf{DY}}^n(\otimes_{k=1}^n [\mathsf{V}_k], \otimes_{k=1}^n [\mathsf{V}_k])$$

is compatible with the face maps $\{d_i^n\}_{i=0}^{n+1}$. The case $i = 0, n+1$ is easily checked. To check the compatibility with d_1^n, it suffices to consider the case $n = 1$. Let $\phi \in \underline{\mathsf{LBA}}([p],[q])$. We need to check the equality of

$$d_1^1 \circ \mathsf{a}^1(\phi) = \pi^{(q)}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]} \circ \phi \otimes \mathrm{id}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]} \circ \pi^{*(p)}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]}$$

and

$$\mathsf{a}^2 \circ d_1^1(\phi) = \sum_{\substack{\underline{p},\underline{q}\in\mathbb{N}^2 \\ |\underline{p}|=p, |\underline{q}|=q}} \pi^{(\underline{q})}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]} \circ d_1^1(\phi)_{\underline{p},\underline{q}} \otimes \mathrm{id}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]} \circ \pi^{*(\underline{p})}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]}$$

where $d_1^1(\phi)_{\underline{p},\underline{q}} \in \underline{\mathsf{LBA}}(T^{\underline{p}}[1], T^{\underline{q}}[1])$ are the homogeneous components of $d_1^1(\phi) = \Delta \circ \phi \circ m$, and m, Δ are the multiplication and comultiplication of $T[1]$.

The equality now follows from the identities

$$\bigoplus_{\underline{p}:|\underline{p}|=p} m \otimes \mathrm{id}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]} \circ \pi^{*(\underline{p})}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]} = \pi^{*(p)}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]}$$

$$\bigoplus_{\underline{q}:|\underline{q}|=q} \pi^{(\underline{q})}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]} \circ \Delta \otimes \mathrm{id}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]} = \pi^{(q)}_{[\mathsf{V}_1]\otimes[\mathsf{V}_2]}$$

of maps $[\mathsf{V}_1]\otimes[\mathsf{V}_2]\to[p]\otimes[\mathsf{V}_1]\otimes[\mathsf{V}_2]$ and $[q]\otimes[\mathsf{V}_1]\otimes[\mathsf{V}_2]\to[\mathsf{V}_1]\otimes[\mathsf{V}_2]$ respectively. The first (resp. second) one holds because both sides are the components of the coaction (resp. action) of $T[1]$ on $[\mathsf{V}_1]\otimes[\mathsf{V}_2]$. $\square$

5.10. **A basis of $\mathfrak{U}^1_{\mathsf{DY}}$.** In the following paragraphs, we describe the vector space underlying $\mathfrak{U}^n_{\mathsf{DY}}$, and the convolution action of $\underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n})$ on it in terms of free algebras and Lie algebras, in analogy with 4.3–4.4. We then use this description to prove an analogue of the PBW theorem for $\mathfrak{U}^n_{\mathsf{DY}}$ in 5.17.

Let $\pi^{(N)}:[N]\otimes[\mathsf{V}_1]\to[\mathsf{V}_1]$ (resp. $\pi^{*(N)}:[\mathsf{V}_1]\to[N]\otimes[\mathsf{V}_1]$) be the Nth iterated action (resp. coaction) on $[\mathsf{V}_1]$.

Proposition. *The endomorphisms of $[\mathsf{V}_1]\in\underline{\mathsf{DY}}^1$ given by*

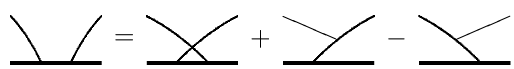
$$r^{\sigma}_{N,N}=\pi^{(N)}\circ\sigma\otimes\mathrm{id}_{[V_1]}\circ\pi^{*(N)}=\sigma\cdot\mathrm{id}_{[V_1]}$$

for $N\geqslant 0$ and $\sigma\in\mathfrak{S}_N$, form a basis of $\mathfrak{U}_{\mathsf{DY}}=\mathsf{End}_{\underline{\mathsf{DY}}^1}([\mathsf{V}_1])$.

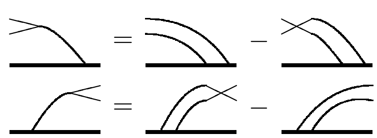
Proof. We represent $\mathrm{id}_{[1]}$ with a line and $\mathrm{id}_{[\mathsf{V}_1]}$ with a bold line. The morphisms μ,δ,π,π^* in $\underline{\mathsf{DY}}^1$ are then represented by the diagrams

and

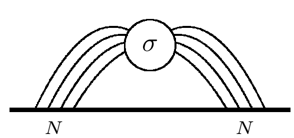
which are read from left to right. A non–trivial endomorphism of $[\mathsf{V}_1]$ is represented as a linear combination of oriented diagrams, necessarily starting with a coaction and ending with an action. The compatibility relation (2.4)



allows to reorder π and π^*, moving every coaction before any action. The cocycle condition (2.14) allows to reorder brackets and cobrackets as in $\underline{\mathsf{LBA}}$. Finally, the relations (2.2), (2.3)



allow to remove from the graph every μ and every δ involved. It follows that every endomorphism of $[\mathsf{V}_1]$ is a linear combination of the elements $r^{\sigma}_{N,N}$ given by



where $N \geqslant 0$ and $\sigma \in \mathfrak{S}_N$. These morphisms are linearly independent in $\underline{\mathsf{DY}}^1$, since they are on the free Drinfeld–Yetter module constructed over the comodule $[\mathsf{V}_1]$, following an argument similar to 4.1. $\square$

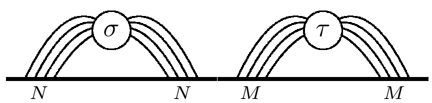
Remark. Under the map $\rho_{\mathfrak{b}} : \mathfrak{U}_{\mathsf{DY}} \to \mathcal{U}_{\mathfrak{b}}$, the basis element $r^{\sigma}_{N,N}$ maps to the interlaced Nth power of the normally ordered Casimir operator of $\mathfrak{g}_{\mathfrak{b}}$ given by

$$\kappa^{\sigma}_N = \sum_{i_1,\dots,i_N} b_{i_{\sigma(1)}} b_{i_{\sigma(2)}} \cdots b_{i_{\sigma(N)}} \cdot b^{i_N} \cdots b^{i_2} b^{i_1}$$

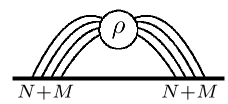
Remark. Proposition 5.10 yields in particular an isomorphism of vector spaces

$$\mathsf{End}_{\underline{\mathsf{DY}}^1}([\mathsf{V}_1]) \simeq \bigoplus_{N \geqslant 0} \mathsf{k}\mathfrak{S}_N \tag{5.1}$$

mapping $r^{\sigma}_{N,N}$ to $\sigma \in \mathfrak{S}_N$. It is clear from the description above that the multiplication in $\mathsf{End}_{\underline{\mathsf{DY}}^1}([\mathsf{V}_1])$ is $\mathbb{N}$–graded, as the normal ordering on the product of two elements of the basis preserves the total number of strings. Namely, for any $N, M > 0$, $\sigma \in \mathfrak{S}_N$, $\tau \in \mathfrak{S}_M$, one gets



$= \sum_{\rho \in \mathfrak{S}_{N+M}} c^{\rho}_{\sigma,\tau}$



for some $c^{\rho}_{\sigma,\tau} \in \mathbb{Z}$. It seems an interesting problem to determine the structure constants $c^{\rho}_{\sigma,\tau}$ explicitly.

5.11. **Convolution product on $\mathfrak{U}_{\mathsf{DY}}$.** Under the isomorphism (5.1), the exterior product of permutations $\otimes : \mathfrak{S}_N \times \mathfrak{S}_M \to \mathfrak{S}_{N+M}$ gives rise to a convolution product $\star$ on $\mathfrak{U}^1_{\mathsf{DY}}$, defined on the basis elements by

$$r^{\sigma}_{N,N} \star r^{\tau}_{M,M} = r^{\sigma\otimes\tau}_{N+M,N+M}$$

Pictorially, the product $\star$ corresponds to the encapsulation of $r^{\tau}_{M,M}$ inside $r^{\sigma}_{N,N}$. In particular, the action of $\underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n})$ commutes with convolution in $\mathfrak{U}_{\mathsf{DY}}$ on the right, *i.e.,*

$$\phi \cdot (X \star Y) = (\phi \cdot X) \star Y$$

for any $\phi \in \underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n})$ and $X, Y \in \mathfrak{U}_{\mathsf{DY}}$. It follows that the action is given by left convolution with $\mathsf{a}^1(\phi) = \phi \cdot \mathrm{id}_{[V]}$, and that a^1 is a morphism with respect to convolution.

In terms of the interlaced Casimir operators of $\mathcal{U}_{\mathfrak{b}}$, one has

$$\begin{aligned}\rho_{\mathfrak{b}}(r^{\sigma}_{N,N} \star r^{\tau}_{M,M}) &= \rho_{\mathfrak{b}}(r^{\sigma\otimes\tau}_{N+M,N+M}) \\ &= \sum_{\substack{i_1,\dots,i_N \\ j_1,\dots,j_M}} \left(b_{i_{\sigma(1)}} \cdots b_{i_{\sigma(N)}}\right) \cdot \left(b_{j_{\tau(1)}} \cdots b_{j_{\tau(M)}}\right) \cdot \left(b^{j_M} \cdots b^{j_1}\right) \cdot \left(b^{i_N} \cdots b^{i_1}\right)\end{aligned}$$

The product $\star$ can therefore be thought of as a PROPic analogue of the polarised multiplication on $U\mathfrak{g}_{\mathfrak{b}} \simeq U\mathfrak{b} \otimes U\mathfrak{b}^*$ given by $(x \otimes y) \star (x' \otimes y') = (x \cdot x') \otimes (y' \cdot y)$. Thus, $\mathfrak{U}_{\mathsf{DY}}$ is endowed with two distinct products: the canonical one coming from its definition as $\mathrm{End}([V_1])$, which corresponds to the usual product on $U\mathfrak{g}_{\mathfrak{b}}$, and the convolution product $\star$, which is defined in terms of the basis $r^{\sigma}_{N,N}$ and corresponds to the polarised product on $U\mathfrak{g}_{\mathfrak{b}}$.

5.12. **A basis of $\mathfrak{U}^n_{\mathsf{DY}}$, $n>1$.** The description of the morphisms in $\underline{\mathsf{DY}}^n$ is similar to the case $n=1$. For any $N\in\mathbb{N}$ and $\underline{N}=(N_1,\dots,N_n)\in\mathbb{N}^n$ such that $|\underline{N}|=N$, let

$$\pi^{(\underline{N})}:[N]\otimes\bigotimes_{k=1}^n[\mathsf{V}_k]\to\bigotimes_{k=1}^n[\mathsf{V}_k]\qquad\text{and}\qquad\pi^{*(\underline{N})}:\bigotimes_{k=1}^n[\mathsf{V}_k]\to[N]\otimes\bigotimes_{k=1}^n[\mathsf{V}_k]$$

be the ordered composition of N_i actions (resp. coactions) on $[\mathsf{V}_i]$.

Proposition. *The endomorphisms of $[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]$ given by*

$$r^{\sigma}_{\underline{N},\underline{N}'}=\pi^{(\underline{N})}\circ\sigma\otimes\mathrm{id}\circ\pi^{*(\underline{N}')}$$

where $N\geqslant 0$, $\underline{N},\underline{N}'\in\mathbb{N}^n$ are such that $|\underline{N}|=N=|\underline{N}'|$, and $\sigma\in\mathfrak{S}_N$, form a basis of $\mathfrak{U}^n_{\mathsf{DY}}=\mathsf{End}_{\underline{\mathsf{DY}}^n}(\otimes_{k=1}^n[\mathsf{V}_k])$.

As in the case of $n=1$, the basis $r^{\sigma}_{\underline{N},\underline{N}'}$ gives rise to a convolution product on $\mathfrak{U}^n_{\mathsf{DY}}$ given by

$$r^{\sigma}_{\underline{N},\underline{N}}\star r^{\tau}_{\underline{N}',\underline{N}'}=r^{\sigma\otimes\tau}_{\underline{N}+\underline{N}',\underline{N}+\underline{N}'}$$

providing a PROPic analogue of the polarised multiplication in $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}$. One checks easily that, with respect to $\star$, the map a^n defined in 5.9 is a morphism of algebras.

5.13. **Cosimplicial structure and basis elements.** The cosimplicial structure of $\mathfrak{U}^n_{\mathsf{DY}}$ introduced in 5.7 is defined on the elements $r^{\sigma}_{\underline{N},\underline{N}'}$ as follows. The degeneration map $\mathcal{E}^i_n:\mathfrak{U}^n_{\mathsf{DY}}\to\mathfrak{U}^{n-1}_{\mathsf{DY}}$ is given by

$$\mathcal{E}^i_n(r^{\sigma}_{\underline{N},\underline{N}'})=\begin{cases}r^{\sigma}_{\underline{N}_{\widehat{i}},\underline{N}'_{\widehat{i}}} & \text{if } N_i=0=M_i\\ 0 & \text{otherwise}\end{cases}$$

where $\underline{N}_{\widehat{i}}$ is obtained from $\underline{N}$ by removing N_i. The face map $\Delta^n_i:\mathfrak{U}^n_{\mathsf{DY}}\to\mathfrak{U}^{n+1}_{\mathsf{DY}}$ is given by

$$\Delta^n_i(r^{\sigma}_{\underline{N},\underline{N}'})=\sum_{\substack{p=0,\dots,N_i\\ q=0,\dots N'_i}}\ \sum_{\substack{\tau\in\mathfrak{G}(N_i,p)\cup\{\mathrm{id}\}\\ \tau'\in\mathfrak{G}(N'_i,N'_i-q)\cup\{\mathrm{id}\}}} r^{(\tau')^{-1}\circ\sigma\circ\tau}_{\underline{N}_p,\underline{N}'_q}$$

where $\underline{N}_p=(N_1,\dots,N_{i-1},p,N_i-p,N_{i+1},\dots,N_n)$ and $\mathfrak{G}(N_i,p)\subset\mathfrak{S}_N$ is the set of permutations τ acting on $(1,\dots,N_{i-1},N_{i-1}+N_i+1,\dots N)$ as the identity and on $(N_{i-1}+1,\dots,N_{i-1}+N_i)$ as a Grassmannian permutations with a unique descent at $N_{i-1}+p$[10]. Similarly for $\underline{N}_q$ and $\mathfrak{G}(N'_i-q,N'_i)$. Note that the appearance of the corrections τ,τ' are due to the prescribed order of actions and coactions on the basis elements $r^{\sigma}_{\underline{N},\underline{N}'}$.

Moreover, one can verify by direct inspection that the face and degenerations maps are morphisms of convolution algebras.

[10]Recall that a Grassmannian permutation is a permutation $\tau\in\mathfrak{S}_N$ with a unique descent. In other words there exists $k\in\{1,\dots,N-1\}$ such that $\tau(i)<\tau(i+1)$ if $i\neq k$ and $\tau(k)>\tau(k+1)$.

5.14. **Identification with free algebras.** Let $\mathcal{A}_N$ be the free algebra in N variables, and, for any $n \geq 1$, denote by $(\mathcal{A}_N^{\otimes n})_{\delta_N} \subset \mathcal{A}_N$ the subspace spanned by elements of degree one in each variable. The symmetric group $\mathfrak{S}_N$ acts diagonally on $(\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}$ by simultaneous permutation of the variables. The corresponding space of coinvariants $\big((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N}$ has the following basis. For any $\underline{N}, \underline{N}' \in \mathbb{N}^n$ such that $|\underline{N}| = N = |\underline{N}'|$ and $\sigma \in \mathfrak{S}_N$, define $x_{\underline{N}}, y_{\sigma(\underline{N}')} \in (\mathcal{A}_N^{\otimes n})_{\delta_N}$ by

$$x_{\underline{N}} = x_1 \cdots x_{N_1} \otimes x_{N_1+1} \cdots x_{N_1+N_2} \otimes \cdots \otimes x_{N_1+\cdots+N_{n-1}+1} \cdots x_N$$
$$y_{\sigma(\underline{N}')} = y_{\sigma(1)} \cdots y_{\sigma(N_1')} \otimes \cdots \otimes y_{\sigma(N_1'+\cdots+N_{n-1}'+1)} \cdots y_{\sigma(N)}$$

Then, $\{x_{\underline{N}} \otimes y_{\sigma(\underline{N}')}\}_{\underline{N},\underline{N}',\sigma}$ is a basis of $\big((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N}$.

The following is an immediate consequence of Proposition 5.12.

Corollary. *The linear map*

$$\xi_{\mathsf{DY}}^n : \mathfrak{U}_{\mathsf{DY}}^n \to \bigoplus_{N \geqslant 0} \big((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N}$$

given by

$$\xi_{\mathsf{DY}}^n(r_{\underline{N},\underline{N}'}^{\sigma}) = x_{\underline{N}} \otimes y_{\widetilde{\sigma}(\underline{N}')} \tag{5.2}$$

where $\widetilde{\sigma} = \sigma^{-1} \circ \tau$ and $\tau \in \mathfrak{S}_N$, such that $\tau(i) = N - i$, is an isomorphism of vector spaces.[11]

5.15. **Module structure on coinvariants.** The space of coinvariants

$$\mathcal{A}^n := \bigoplus_{N \geqslant 0} \big((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N}$$

is an associative algebra, with product map in degree (M, N)

$$\big(\mathcal{A}_M^{\otimes n} \otimes \mathcal{A}_M^{\otimes n}\big)_{\mathfrak{S}_M} \otimes \big(\mathcal{A}_N^{\otimes n} \otimes \mathcal{A}_N^{\otimes n}\big)_{\mathfrak{S}_N} \to \big(\mathcal{A}_{M+N}^{\otimes n} \otimes \mathcal{A}_{M+N}^{\otimes n}\big)_{\mathfrak{S}_{M+N}}$$

given by the formula

$$(x_{\underline{M}} \otimes y_{\widetilde{\sigma}(\underline{M}')}) \star (x_{\underline{N}} \otimes y_{\widetilde{\tau}(\underline{N}')}) = (x_{\underline{M}} \cdot x_{\underline{N}}) \otimes (y_{\widetilde{\tau}(\underline{N}')} \cdot y_{\widetilde{\sigma}(\underline{M}')}) \tag{5.3}$$

where $x_{\underline{M}} \cdot x_{\underline{N}}$ and $y_{\widetilde{\tau}(\underline{N}')} \cdot y_{\widetilde{\sigma}(\underline{M}')}$ are identified with elements in $\mathcal{A}_{M+N}^{\otimes n}$. Note that, under the identification provided by ξ_{DY}^n, (5.3) reads

$$\xi_{\mathsf{DY}}^n(r_{M,M}^{\sigma}) \star \xi_{\mathsf{DY}}^n(r_{N,N}^{\tau}) = \xi_{\mathsf{DY}}^n(r_{M+N,M+N}^{\sigma\otimes\tau}) = \xi_{\mathsf{DY}}^n(r_{M,M}^{\sigma} \star r_{N,N}^{\tau})$$

The formula (5.3) is easily adapted to define an algebra structure on

$$\mathcal{T}^n := \bigoplus_{N \geqslant 0} \big((T\mathcal{L}_N^{\otimes n})_{\delta_N} \otimes (T\mathcal{L}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N}$$

In particular, the linear surjection $\mathsf{p}^n : \mathcal{T}^n \to \mathcal{A}^n$, defined componentwise by the quotient map $T\mathcal{L}_N \to U\mathcal{L}_N = \mathcal{A}_N$ for the free Lie algebra $\mathcal{L}_N$, is an algebra map and induces on $\mathcal{A}^n$ a natural structure of $\mathcal{T}^n$–module.

[11] The involution τ_N is required because of the contravariance of the expression (2.13) with respect to the Lie polynomial Q, and to ensure the commutativity of the diagram in Theorem 5.17.

5.16. **Identification with free Lie algebras.** Let

$$\xi^n_{\mathsf{LBA}} : \mathcal{L}^n := \underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \to \mathcal{T}^n$$

be the isomorphism of vector spaces given by Proposition 4.3. One checks by direct inspection that ξ^n_{LBA} is a morphism of algebras, with respect to the convolution product on $\mathcal{L}^n$ and the multiplication on $\mathcal{T}^n$ defined in 5.15, *i.e.,* $\xi^n_{\mathsf{LBA}}(\phi \star \psi) = \xi^n_{\mathsf{LBA}}(\phi) \star \xi^n_{\mathsf{LBA}}(\psi)$ for any $\phi, \psi \in \mathcal{L}^n$. Therefore, through ξ^n_{LBA}, we obtain a convolution action of $\mathcal{L}^n$ on $\mathcal{A}^n$, *i.e.,* we set

$$\phi \cdot (x_{\underline{N}} \otimes y_{\sigma(\underline{N'})}) = (\mathsf{p}^n \circ \xi^n_{\mathsf{LBA}})(\phi) \cdot (x_{\underline{N}} \otimes y_{\sigma(\underline{N'})})$$

for any $\phi \in \mathcal{L}^n$.

Proposition. *The isomorphism $\xi^n_{\mathsf{DY}} : \mathfrak{U}^n_{\mathsf{DY}} \to \mathcal{A}^n$ given by (5.2) intertwines the convolution actions of $\mathcal{L}^n$, that is satisfies*

$$\xi^n_{\mathsf{DY}}(\phi \cdot X) = (\mathsf{p}^n \circ \xi^n_{\mathsf{LBA}})(\phi) \cdot \xi^n_{\mathsf{DY}}(X)$$

for any $\phi \in \mathcal{L}^n$ and $X \in \mathfrak{U}^n_{\mathsf{DY}}$.

Proof. Assume for simplicity that $n = 1$. The proof for $n > 1$ is identical.

Let $P_1 \otimes \cdots \otimes P_p \in \mathcal{L}_N^{\otimes p}$ be an element of degree δ_N, and $\mu_{P_1 \otimes \cdots \otimes P_p} \in \underline{\mathsf{LA}}([N], [p])$ the element corresponding to it by Lemma 4.2. In $\mathcal{A}_N = U\mathcal{L}_N$, the product $P_1 \cdots P_p$ corresponds to an element $\sigma_{P_1 \cdots P_p} \in (\mathcal{A}_N)_{\delta_N} \simeq \mathsf{k}\mathfrak{S}_N$, which, by (2.2), satisfies the following relation in $\underline{\mathsf{DY}}^1([N] \otimes [\mathsf{V}_1], [\mathsf{V}_1])$

$$\pi^{(p)} \circ \mu_{P_1 \otimes \cdots \otimes P_p} \otimes X = \pi^{(N)} \circ \sigma_{P_1 \cdots P_p} \otimes X \tag{5.4}$$

For example, if $p = 1$, $N = 2$, and $P \in \mathcal{L}_2$ is the element $[x_1, x_2]$, then $\mu_P : [2] \to [1]$ is the Lie bracket and, by (2.2)

$$\pi \circ \mu_P = \pi \circ (\mathrm{id} \otimes \pi) - \pi \circ (\mathrm{id} \otimes \pi) \circ (1\,2) = \pi^{(2)} \circ \sigma_P$$

with $\sigma_P = \mathrm{id} - (1\,2)$. Dually, for any $Q_1 \otimes \cdots \otimes Q_q \in \mathcal{L}_N^{\otimes q}$ of degree δ_N, there are elements

$$\delta_{Q_1 \otimes \cdots \otimes Q_q} \in \underline{\mathsf{LCA}}([q], [N]) \qquad \text{and} \qquad \widetilde{\sigma}_{Q_1 \cdots Q_q} \in \mathsf{k}\mathfrak{S}_N$$

such that the following holds in $\underline{\mathsf{DY}}^1([\mathsf{V}_1], [N] \otimes [\mathsf{V}_1])$

$$\delta_{Q_1 \otimes \cdots \otimes Q_q} \otimes X \circ \pi^{*(q)} = \widetilde{\sigma}_{Q_1 \cdots Q_q} \otimes X \circ \pi^{*(N)} \tag{5.5}$$

The commutativity of the diagram then follows easily. Namely, assume that $X = r^{\sigma}_{M,M}$ for some $M > 0$ and $\sigma \in \mathfrak{S}_M$. In particular, we have $\xi_{\mathsf{DY}}(X) = (x_1 \cdots x_M) \otimes (y_{\widetilde{\sigma}(1)} \cdots y_{\widetilde{\sigma}(M)}) =: Q_X \otimes P_X$ and

$$(\mathsf{p} \circ \xi_{\mathsf{LBA}})(\mu_{P_1 \otimes \cdots \otimes P_p} \circ \delta_{Q_1 \otimes \cdots \otimes Q_q}) = (Q_1 \cdots Q_q) \otimes (P_1 \cdots P_p)$$

Then, by (5.4) and (5.5),

$$\begin{aligned}
(\mu_{P_1 \otimes \cdots \otimes P_p} \circ \delta_{Q_1 \otimes \cdots \otimes Q_q}) \cdot r^{\sigma}_{M,M} &= \pi^{(p)} \circ \mu_{P_1 \otimes \cdots \otimes P_p} \circ \delta_{Q_1 \otimes \cdots \otimes Q_q} \otimes r^{\sigma}_{M,M} \circ \pi^{*(q)} \\
&= \pi^{(N)} \circ \sigma_{P_1 \cdots Pp} \circ (\sigma_{Q_1 \cdots Q_q} \circ \tau_N) \otimes r^{\sigma}_{M,M} \circ \pi^{*(N)} \\
&= \pi^{(N+M)} \circ \sigma_{P_1 \cdots Pp} \circ \widetilde{\sigma}_{Q_1 \cdots Q_q} \otimes \sigma \circ \pi^{*(N+M)}
\end{aligned}$$

which, under ξ_{DY}, corresponds precisely to the element $(Q_1 \cdots Q_q) \cdot Q_X \otimes P_X \cdot (P_1 \cdots P_p)$ in $((\mathcal{A}_{N+M})_{\delta_{N+M}} \otimes (\mathcal{A}_{N+M})_{\delta_{N+M}})_{\mathfrak{S}_{N+M}}$. Therefore

$$\begin{aligned}\xi_{\mathsf{DY}}((\mu_{P_1\otimes\cdots\otimes P_p} \circ \delta_{Q_1\otimes\cdots\otimes Q_q}) \cdot r^{\sigma}_{M,M}) &= \\ &= (\mathsf{p} \circ \xi_{\mathsf{LBA}})(\mu_{P_1\otimes\cdots\otimes P_p} \circ \delta_{Q_1\otimes\cdots\otimes Q_q}) \cdot \xi_{\mathsf{DY}}(r^{\sigma}_{M,M})\end{aligned}$$

and the result follows. □

Applying the result to $X = \mathrm{id}_{\otimes_{k=1}^n [\mathsf{V}_k]}$, yields the following

Corollary. *The following is a commutative diagram of convolution algebras.*

$$\begin{tikzcd}
\underline{\mathsf{DY}}^n(\bigotimes_{k=1}^n [\mathsf{V}_k], \bigotimes_{k=1}^n [\mathsf{V}_k]) \arrow[r, "\xi^n_{\mathsf{DY}}"] & \bigoplus_{N\geqslant 0} \big((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N} \\
\underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \arrow[u, "\mathsf{a}^n"] \arrow[r, "\xi^n_{\mathsf{LBA}}"'] & \bigoplus_{N\geqslant 0} \big((T\mathcal{L}_N^{\otimes n})_{\delta_N} \otimes (T\mathcal{L}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N} \arrow[u, "\mathsf{p}^n"']
\end{tikzcd}$$

5.17. **PBW theorem for $\mathfrak{U}^n_{\mathsf{DY}}$.** Let

$$\mathsf{Sym} : \underline{\mathsf{LBA}}(\widehat{S}[1]^{\otimes n}, S[1]^{\otimes n}) \to \underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n})$$

be the map (4.2). The following result shows that the composition $\mathsf{a} \circ \mathsf{Sym}$ can be thought of as the symmetrisation map $S\mathfrak{l} \to U\mathfrak{l}$ for a Lie algebra $\mathfrak{l}$.

Theorem. *The following is a commutative diagram*

$$\begin{tikzcd}
\underline{\mathsf{DY}}^n(\bigotimes_{k=1}^n [\mathsf{V}_k], \bigotimes_{k=1}^n [\mathsf{V}_k]) \arrow[r] & \bigoplus_{N\geqslant 0} \big((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N} \\
\underline{\mathsf{LBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \arrow[u, "\mathsf{a}^n"] \arrow[r] & \bigoplus_{N\geqslant 0} \big((T\mathcal{L}_N^{\otimes n})_{\delta_N} \otimes (T\mathcal{L}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N} \arrow[u, "\mathsf{p}^n"'] \\
\underline{\mathsf{LBA}}(\widehat{S}[1]^{\otimes n}, S[1]^{\otimes n}) \arrow[u, "\mathsf{Sym}"] \arrow[r] & \bigoplus_{N\geqslant 0} \big((S\mathcal{L}_N^{\otimes n})_{\delta_N} \otimes (S\mathcal{L}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N} \arrow[u, "\mathsf{Sym}\otimes\mathsf{Sym}"']
\end{tikzcd}$$

where the right vertical arrows are the symmetrisation map $S\mathcal{L}_N \to T\mathcal{L}_N$ and quotient map $T\mathcal{L}_N \to U\mathcal{L}_N = \mathcal{A}_N$ for the Lie algebra $\mathcal{L}_N$.

Moreover, the map $\mathsf{a} \circ \mathsf{Sym}$ is an isomorphism of cosimplicial spaces.

Proof. The commutativity of the diagram follows from Lemma 4.4 and Proposition 5.16. The fact that $\mathsf{a} \circ \mathsf{Sym}$ is an isomorphism then follows from the PBW Theorem for the Lie algebra $\mathcal{L}_N$, and the fact that it is compatible with the cosimplicial structure from Proposition 5.9. □

5.18. **PBW conjecture for $\underline{\mathsf{DY}}^n$.** Let $\underline{\mathsf{SDY}}^n$ be the colored PROP generated by an $\underline{\mathsf{LBA}}$–module $([1],\mu,\delta)$ and objects $[\mathsf{V}_k]$ endowed with maps $\pi_{[\mathsf{V}_k]}:[1]\otimes[\mathsf{V}_k]\to[\mathsf{V}_k]$, $\pi^*_{[\mathsf{V}_k]}:[\mathsf{V}_k]\to[1]\otimes[\mathsf{V}_k]$, $k=1,\dots,n$, satisfying the relations

$$\pi_{[\mathsf{V}_k]}\circ\mathrm{id}_{[1]}\otimes\pi_{[\mathsf{V}_k]}\circ(\mathrm{id}_{[2]}-(1\,2))\otimes\mathrm{id}_{[\mathsf{V}_k]}=0$$

$$(\mathrm{id}_{[2]}-(1\,2))\otimes\mathrm{id}_{[\mathsf{V}_k]}\circ\mathrm{id}\otimes\pi^*_{[\mathsf{V}_k]}\circ\pi^*_{[\mathsf{V}_k]}=0$$

$$\pi^*_{[\mathsf{V}_k]}\circ\pi_{[\mathsf{V}_k]}=\mathrm{id}\otimes\pi_{[\mathsf{V}_k]}\circ(1\,2)\circ\mathrm{id}\otimes\pi^*_{[\mathsf{V}_k]}$$

Thus, $\underline{\mathsf{SDY}}^n$ encodes a Lie bialgebra $\mathfrak{b}$, together with n Drinfeld–Yetter modules over the underlying vector space of $\mathfrak{b}$ endowed with trivial bracket and cobracket.

The PROP $\underline{\mathsf{DY}}^n$ (resp. $\underline{\mathsf{SDY}}^n$) is $\mathbb{N}$–filtered (resp. graded) by $\deg(\delta)=0=\deg(\mu)$, and $\deg\pi_{[\mathsf{V}_k]}=1=\deg\pi^*_{[\mathsf{V}_k]}$. Moreover, there is a canonical filtered functor $\underline{\mathsf{SDY}}^n\to\mathsf{gr}(\underline{\mathsf{DY}}^n)$ which is the identity on objects and is easily seen to be full. It is natural to conjecture the following results which, together, extend Theorem 5.17.

Conjecture.

(1) *The functor* $\underline{\mathsf{SDY}}^n\to\mathsf{gr}(\underline{\mathsf{DY}}^n)$ *is faithful, and therefore an isomorphism of* PROP*s.*

(2) *The map* $\mathsf{a}\circ\mathsf{Sym}:\underline{\mathsf{LBA}}(\widehat{S}[1]^{\otimes n},S[1]^{\otimes n})\to\underline{\mathsf{SDY}}^n(\bigotimes_{k=1}^n[\mathsf{V}_k],\bigotimes_{k=1}^n[\mathsf{V}_k])$ *is an isomorphism.*

5.19. **Cohomology of $\mathfrak{U}_{\mathsf{DY}}$.**

Theorem. *The map* $\mathsf{a}\circ\mathsf{Sym}$ *induces an isomorphism*

$$H^n(\mathfrak{U}^\bullet_{\mathsf{DY}},d_H)\simeq\bigoplus_{j=0}^n\underline{\mathsf{LBA}}\left(\wedge^j[1],\wedge^{n-j}[1]\right)$$

In particular, $H^0(\mathfrak{U}^\bullet_{\mathsf{DY}},d_H)=\mathsf{k}$ *and* $H^1(\mathfrak{U}^\bullet_{\mathsf{DY}},d_H)=0$.

Proof. By Theorem 5.17, $\mathsf{a}\circ\mathsf{Sym}$ is an isomorphism of cosimplicial spaces. The result then follows from Proposition 3.12 applied to the PROP $\underline{\mathsf{P}}=\underline{\mathsf{LBA}}$. Namely, we have

$$\underline{\mathsf{LBA}}(\widehat{S}[1]^{\otimes\bullet},S[1]^{\otimes\bullet})=\underline{\mathsf{LBA}}((S^*)^{\otimes\bullet}[1],S^{\otimes\bullet}[1])=\underline{\mathsf{LBA}}_{\mathsf{Sch}}(S^{\otimes\bullet}\boxtimes S^{\otimes\bullet})$$

where the first equality relies on the equality of Schur functors $\widehat{S}=S^*$, and the fact that the cosimplicial structure on $S[1]^{\otimes\bullet}$ (resp. the simplicial structure on $\widehat{S}[1]^{\otimes\bullet}$) is induced by that on the Schur functors $S^{\otimes\bullet}$ (resp. $S^{*\otimes\bullet}$), and the second one from (3.15). The result now follows from Proposition 3.12, and the equality of Schur functors $(\wedge^n)^*=\wedge^n$ for any $n\geqslant 0$. □

Remark. Theorem 5.19 can also be obtained via Lemma 4.4 from the fact that the diagram in 5.17 is one of cosimplicial spaces, and the standard computation of the Hochschild cohomology of a symmetric algebra. The proof via Schur bifunctors given above yields a more uniform answer for the refinements of the PROP $\underline{\mathsf{LBA}}$ introduced in Sections 6 and 9. Note, however that it still depends on the PBW Theorem for $\mathfrak{U}^\bullet_{\mathsf{DY}}$, which is obtained from the identification of $\underline{\mathsf{DY}}^n(\bigotimes_{k=1}^n[\mathsf{V}_k],\bigotimes_{k=1}^n[\mathsf{V}_k])$ (resp. $\underline{\mathsf{LBA}}(\widehat{S}[1]^{\otimes n},S[1]^{\otimes n})$) with free algebras (resp. Lie algebras).

5.20. **Explicit description of $H^n(\mathfrak{U}_{\mathsf{DY}}^\bullet, d_H)$.** The cohomology of $\mathfrak{U}_{\mathsf{DY}}^\bullet$ can be described more explicitly, in the spirit of 3.12. Denote by $\widetilde{\iota}_{n,j}$ the inclusion

$$\underline{\mathsf{LBA}}(\wedge^j[1], \wedge^{n-j}[1]) \to \underline{\mathsf{LBA}}(\widehat{S}[1]^{\otimes n}, S[1]^{\otimes n})$$

defined by (3.17). We first observe that in $\underline{\mathsf{DY}}^1$

$$\pi_{T[1]} \circ \iota_0 \otimes \mathrm{id}_{[\mathsf{V}_1]} = \mathrm{id}_{[\mathsf{V}_1]} \qquad \text{and} \qquad \pi_{T[1]} \circ \iota_1 \otimes \mathrm{id}_{[\mathsf{V}_1]} = \pi_{[\mathsf{V}_1]}$$

and dually

$$\iota_0{}^* \otimes \mathrm{id}_{[\mathsf{V}_1]} \circ \pi^*_{\widehat{T}[1]} = \mathrm{id}_{[\mathsf{V}_1]} \qquad \text{and} \qquad \iota_1{}^* \otimes \mathrm{id}_{[\mathsf{V}_1]} \circ \pi^*_{\widehat{T}[1]} = \pi^*_{[\mathsf{V}_1]}$$

It follows that the inclusion

$$\iota_{n,j} : \underline{\mathsf{LBA}}(\wedge^j[1], \wedge^{n-j}[1]) \to \underline{\mathsf{DY}}^n(\otimes_{k=1}^n[\mathsf{V}_k], \otimes_{k=1}^n[\mathsf{V}_k])$$

where $\iota_{n,j} = \mathsf{a} \circ \mathrm{Sym} \circ \widetilde{\iota}_{n,j}$, sends a morphism $\phi \in \underline{\mathsf{LBA}}(\wedge^j[1], \wedge^{n-j}[1])$ to

$$\iota_{n,j}(\phi) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \pi_{\overline{J_\sigma}} \circ \phi \circ \pi^*_{J_\sigma}$$

where $J_\sigma = \{\sigma(1), \dots, \sigma(j)\}$, $\overline{J}_\sigma$ is its complement, $\pi_{\overline{J}_\sigma}$ denotes the ordered action of $[n-j] = [1]^{\otimes n-j}$ on the components $[\mathsf{V}_k]$, $k \in \overline{J}_\sigma$, and $\pi^*_{J_\sigma}$ the ordered coaction of $[j] = [1]^{\otimes j}$ on the components $[\mathsf{V}_k]$, $k \in J_\sigma$. For example, for $n = 2$, we have

$$\iota_{2,1}(\mathrm{id}_{[1]}) = \frac{1}{2}\left(\pi_{[\mathsf{V}_2]} \circ (1\,2) \circ \pi^*_{[\mathsf{V}_1]} - \pi_{[\mathsf{V}_1]} \circ (1\,2) \circ \pi^*_{[\mathsf{V}_2]}\right) = \frac{1}{2}\left(r_{[\mathsf{V}_1],[\mathsf{V}_2]} - r_{[\mathsf{V}_2],[\mathsf{V}_1]}\right)$$

i.e., the antisymmetric r–matrix corresponds to the identity in $\underline{\mathsf{LBA}}([1],[1])$.

Thus, the image of $\underline{\mathsf{LBA}}(\wedge^j[1], \wedge^{n-j}[1])$ inside $\underline{\mathsf{DY}}^n(\otimes_{k=1}^n[\mathsf{V}_k], \otimes_{k=1}^n[\mathsf{V}_k])$ consists of linear combinations of arc diagrams with exactly one coaction or one action on each bold line, which are antisymmetric under permutation of the bold lines.

In terms of the identification with free Lie algebras given by Proposition 4.3, the above isomorphism yields

$$H^i(\mathfrak{U}_{\mathsf{DY}}^\bullet, d_H) \cong \bigoplus_{N \geqslant 0} \bigoplus_{j=0}^{i} \left[\left(\wedge^j \mathcal{L}_N\right)_{\delta_N} \otimes \left(\wedge^{i-j} \mathcal{L}_N\right)_{\delta_N}\right]_{\mathfrak{S}_N}$$

Then $H^i(\mathfrak{U}_{\mathsf{DY}}^\bullet, d_H)$ embeds in $\mathfrak{U}_{\mathsf{DY}}^i \simeq \bigoplus_{N \geqslant 0} \left((\mathcal{A}_N^{\otimes i})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes i})_{\delta_N}\right)_{\mathfrak{S}_N}$ via (3.12).

5.21. **Enriquez's universal algebras.** In [10, 11, 12], Enriquez introduced the universal algebras $\{U\mathfrak{G}^n_{\mathrm{univ}}\}_{n \geq 1}$ associated to the PROP $\underline{\mathsf{LBA}}$. As a k–vector space, $U\mathfrak{G}^n_{\mathrm{univ}}$ is defined as the space of coinvariants

$$U\mathfrak{G}^n_{\mathrm{univ}} = \bigoplus_{N \geqslant 0} \left((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\right)_{\mathfrak{S}_N}$$

introduced in 5.14–5.15. The multiplication on $U\mathfrak{G}^n_{\mathrm{univ}}$ is defined by an explicit formula in the basis $\{x_{\underline{N}} \otimes y_{\sigma(\underline{N'})}\}_{\underline{N},\underline{N'},\sigma}$ given in 5.14, and proved to be associative by a lengthy calculation [11, B.1–B.2].[12]

[12] The multiplication on $U\mathfrak{G}^n_{\mathrm{univ}}$ differs from the convolution product discussed in 5.15.

$U\mathfrak{G}_{\text{univ}}$ is universal in the following sense. For any Lie bialgebra $\mathfrak{b}$ with Drinfeld double $\mathfrak{g}_\mathfrak{b} = \mathfrak{b} \oplus \mathfrak{b}^*$ and r–matrix $r_\mathfrak{b} = \sum_{i\in I} b_i \otimes b^i \in \mathfrak{b}\hat{\otimes}\mathfrak{b}^*$, the linear map $\rho_{\mathfrak{g}_\mathfrak{b}} : U\mathfrak{G}_{\text{univ}} \to \widehat{U\mathfrak{g}_\mathfrak{b}}$ given by

$$\rho_{\mathfrak{g}_\mathfrak{b}}(x_1 \cdots x_N \otimes y_{\sigma(1)} \cdots y_{\sigma(N)}) = \sum_{\underline{i}\in I^N} b_{i_1} \cdots b_{i_N} b^{i_{\sigma(1)}} \cdots b^{i_{\sigma(N)}}$$

is an algebra homomorphism. Similarly, for any $n \geq 2$, there is a map $\rho^n_{\mathfrak{g}_\mathfrak{b}} : U\mathfrak{G}^n_{\text{univ}} \to \widehat{U\mathfrak{g}_\mathfrak{b}^{\otimes n}}$ given by

$$\rho^n_{\mathfrak{g}_\mathfrak{b}}(x_{\underline{N}} \otimes y_{\sigma(\underline{N}')}) = \sum_{\underline{i}\in I^N} b_{\underline{N}(\underline{i})} \cdot b^{\sigma(\underline{N}')(\underline{i})}$$

is an algebra homomorphism, where

$$b_{\underline{N}(\underline{i})} = \bigotimes_{k=1}^{n} b_{i_{N_1+\cdots+N_{k-1}+1}} \cdots b_{i_{N+1+\cdots+N_k}}$$

$$b^{\sigma(\underline{N}')(\underline{i})} = \bigotimes_{k=1}^{n} b^{i_{\sigma(N'_1+\cdots+N'_{k-1}+1)}} \cdots b^{i_{\sigma(N'_1+\cdots+N'_k)}}$$

5.22. **The isomorphism $\mathfrak{U}^n_{\mathsf{DY}} \simeq U\mathfrak{G}^n_{\text{univ}}$.** The following result identifies the algebra $U\mathfrak{G}^n_{\text{univ}}$ with $\mathfrak{U}^n_{\mathsf{DY}}$, thereby considerably simplifying the proof of the existence of an algebra structure on $U\mathfrak{G}^n_{\text{univ}}$ given in [11, Appendices B and C].

Let $\xi^n_{\mathsf{DY}} : \mathfrak{U}^n_{\mathsf{DY}} \to U\mathfrak{G}^n_{\text{univ}} = \bigoplus_{N\geqslant 0} \big((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N}$ be the map defined in 5.14.

Proposition.

(1) ξ^n_{DY} *is an isomorphism of cosimplicial spaces.*
(2) *There is a commutative diagram*

$$\begin{array}{ccc} \mathfrak{U}^n_{\mathsf{DY}} & \xrightarrow{\rho^n_\mathfrak{b}} & \mathcal{U}^n_\mathfrak{b} \\ {\scriptstyle \xi^n_{\mathsf{DY}}}\big\downarrow & & \big\| \\ U\mathfrak{G}^n_{\text{univ}} & \xrightarrow[\rho^n_{\mathfrak{g}_\mathfrak{b}}]{} & \widehat{U\mathfrak{g}_\mathfrak{b}^{\otimes n}} \end{array}$$

Proof. (1) The fact that ξ^n_{DY} is an isomorphism was proved in 5.14, and its compatibility with the cosimplicial structure in Lemma 4.4. (2) The commutativity of the diagram follows by direct inspection. $\square$

Remark. It seems very likely that the map ξ^n_{DY} is an algebra homomorphism. This would follow from a detailed inspection of the algebra structure on $U\mathfrak{G}^n_{\text{univ}}$, or from the commutativity of the above diagram if the collection of maps $\rho^n_{\mathfrak{g}_\mathfrak{b}}$ were known to be be injective. In any event, the above proposition shows that $\mathfrak{U}^n_{\mathsf{DY}}$ is an isomorphic replacement of $U\mathfrak{G}^n_{\text{univ}}$ with a more naturally defined multiplication.

6. The universal algebra of a split pair

In this section, we give a relative version of the results of Sections 5 by adapting them to case of a split pair of Lie bialgebras, as defined in [1].

6.1. **The PROP $\underline{\mathsf{PLBA}}$.** Let $(\mathfrak{b},\mathfrak{a})$ be a split pair of Lie bialgebras, *i.e.,* $\mathfrak{b}$ and $\mathfrak{a}$ are Lie bialgebras endowed with Lie bialgebra morphisms

$$\mathfrak{a} \xrightarrow{i_\mathfrak{a}} \mathfrak{b} \xrightarrow{p_\mathfrak{a}} \mathfrak{a}$$

such that $p_\mathfrak{a}\circ i_\mathfrak{a} = \mathrm{id}_\mathfrak{a}$. These maps induce an isometric inclusion of the corresponding Drinfeld doubles $\mathfrak{g}_\mathfrak{a}\subset\mathfrak{g}_\mathfrak{b}$, and a restriction functor $\mathrm{Res}_{\mathfrak{b},\mathfrak{a}} : \mathsf{DY}_\mathfrak{b} \to \mathsf{DY}_\mathfrak{a}$.

Definition. Let $\underline{\mathsf{PLBA}}$ be the colored PROP generated by two Lie bialgebra objects $[\mathfrak{b}],[\mathfrak{a}]$ related by Lie bialgebra morphisms $i_{[\mathfrak{a}]} : [\mathfrak{a}]\to[\mathfrak{b}]$, $p_{[\mathfrak{a}]} : [\mathfrak{b}]\to[\mathfrak{a}]$ such that $p_{[\mathfrak{a}]}\circ i_{[\mathfrak{a}]} = \mathrm{id}_{[\mathfrak{a}]}$.

The kernel $[\mathfrak{m}]$ of the projection $p_{[\mathfrak{a}]}$, is an object of $\underline{\mathsf{PLBA}}$, and $[\mathfrak{b}]$ decomposes as $[\mathfrak{b}] = [\mathfrak{a}]\oplus[\mathfrak{m}]$. $[\mathfrak{m}]$ is an ideal in $[\mathfrak{b}]$, and has a Lie algebra structure. It is also a coideal, but has no natural Lie coalgebra structure.

6.2. **Universal property of $\underline{\mathsf{PLBA}}$.** The following is clear.

Proposition.

(1) *The* PROP $\underline{\mathsf{PLBA}}$ *is endowed with a pair of functors* $\beta,\alpha : \underline{\mathsf{LBA}} \to \underline{\mathsf{PLBA}}$ *given by*

$$\beta[1] = [\mathfrak{b}] \qquad \textit{and} \qquad \alpha[1] = [\mathfrak{a}]$$

The maps $i_{[\mathfrak{a}]}, p_{[\mathfrak{a}]}$ *in* $\underline{\mathsf{PLBA}}$ *induce two natural transformations* $i_\alpha : \alpha \to \beta, p_\alpha : \beta\to\alpha$ *such that* $p_\alpha\circ i_\alpha = \mathrm{id}_\alpha$

$$\begin{tikzcd}[column sep=huge] \underline{\mathsf{LBA}} \arrow[r, bend left=40, "\beta"] \arrow[r, bend right=40, "\alpha"'] \arrow[r, phantom, "i_\alpha \Uparrow\ \Downarrow p_\alpha"] & \underline{\mathsf{PLBA}} \end{tikzcd}$$

(2) $\underline{\mathsf{PLBA}}$ *is universal with respect to property (i): for any tensor category* $\mathcal{C}$ *for which it holds, there is a unique tensor functor* $F : \underline{\mathsf{PLBA}} \to \mathcal{C}$ *such that the following diagram commutes*

$$\begin{tikzcd}[column sep=huge] \underline{\mathsf{LBA}} \arrow[r, bend left=40, "\beta"] \arrow[r, bend right=40, "\alpha"'] \arrow[r, phantom, "i_\alpha \Uparrow\ \Downarrow p_\alpha"] \arrow[rr, bend left=60, "\beta_{\mathcal{C}}"] \arrow[rr, bend right=60, "\alpha_{\mathcal{C}}"'] & \underline{\mathsf{PLBA}} \arrow[r, dashed] \arrow[r, phantom, near start, "i_{\mathcal{C}} \Uparrow\ \Downarrow p_{\mathcal{C}}", yshift=4ex] & \mathcal{C} \end{tikzcd}$$

6.3. **Alternative presentation of $\underline{\mathsf{PLBA}}$.** The following presentation of $\underline{\mathsf{PLBA}}$ is more convenient for computations. Let PLA be the PROP generated by $\mu : [2]\to[1]$ and $\theta_0 : [1]\to[1]$ satisfying the relations (2.10),

$$\theta_0^2 = \theta_0 \qquad \text{and} \qquad \mu\circ(\theta_0\otimes\theta_0) = \theta_0\circ\mu \tag{6.1}$$

Let PLCA be the PROP generated by $\delta : [1]\to[2]$ and $\theta_0 : [1]\to[1]$ satisfying the relations (2.11),

$$\theta_0^2 = \theta_0 \qquad \text{and} \qquad (\theta_0\otimes\theta_0)\circ\delta = \delta\circ\theta_0 \tag{6.2}$$

Let PLBA be the PROP generated by $\mu : [2] \to [1]$, $\delta : [1] \to [2]$, and $\theta_0 : [1] \to [1]$, satisfying the relations (2.10), (2.11), (2.14), (6.1), (6.2). Finally, let $\underline{\mathsf{PLA}}, \underline{\mathsf{PLCA}}, \underline{\mathsf{PLBA}}$ be their corresponding completions.

The two presentations of $\underline{\mathsf{PLBA}}$ are canonically equivalent by sending $[1]$ to $[\mathfrak{b}]$ and the idempotent θ_0 to the composition $i_{[\mathfrak{a}]} \circ p_{[\mathfrak{a}]} : [\mathfrak{b}] \to [\mathfrak{b}]$.

Corollary.

(1) *There is a forgetful functor* $\underline{\mathsf{PLBA}} \to \underline{\mathsf{LBA}}$, *mapping* $[1]_{\underline{\mathsf{PLBA}}}$ *to* $[1]_{\underline{\mathsf{LBA}}}$ *and* θ_0 *to* $\mathrm{id}_{[1]_{\underline{\mathsf{LBA}}}}$.

(2) *There is a forgetful functor* $\underline{\mathsf{PLBA}} \to \underline{\mathsf{LBA}}$, *mapping* $[1]_{\underline{\mathsf{PLBA}}}$ *to* $[1]_{\underline{\mathsf{LBA}}}$ *and* θ_0 *to* 0.

6.4. **Factorisation of morphisms in $\underline{\mathsf{PLBA}}$.** Set $\theta_1 = \mathrm{id} - \theta_0$, and $\mathcal{I} = \{0, 1\}$. The projections

$$\theta_{\underline{i}} = \theta_{i_1} \otimes \cdots \otimes \theta_{i_N}, \quad \underline{i} = (i_1, \ldots, i_N) \in \mathcal{I}^N$$

are a complete family of idempotents in $\underline{\mathsf{PLA}}([N],[N])$ and $\underline{\mathsf{PLCA}}([N],[N])$, *i.e.*,

$$\theta_{\underline{i}} \circ \theta_{\underline{i}'} = \delta_{\underline{i}\underline{i}'} \theta_{\underline{i}} \qquad \text{and} \qquad \sum_{\underline{i} \in \mathcal{I}^N} \theta_{\underline{i}} = \mathrm{id}_{[N]}$$

There is a natural right (resp. left) action of $\mathsf{k}[\mathcal{I}^N]$ on $\underline{\mathsf{PLA}}([N],[q])$ and $\underline{\mathsf{PLCA}}([p],[N])$ given by

$$\phi \cdot f = \sum_{\underline{i} \in \mathcal{I}^N} f(\underline{i})\, \phi \circ \theta_{\underline{i}} \qquad \text{and} \qquad f \cdot \psi = \sum_{\underline{i} \in \mathcal{I}^N} f(\underline{i})\, \theta_{\underline{i}} \circ \psi$$

Set $\Gamma_N = \mathsf{k}[\mathcal{I}^N] \rtimes \mathfrak{S}_N$, where $\sigma \in \mathfrak{S}_N$ acts on $f \in \mathsf{k}[\mathcal{I}^N]$ by $\sigma \cdot f = f \circ \sigma^{-1}$.

Proposition. *The embeddings* $\underline{\mathsf{PLA}}, \underline{\mathsf{PLCA}} \to \underline{\mathsf{PLBA}}$ *induce an isomorphism of* $(\mathfrak{S}_q, \mathfrak{S}_p)$*–bimodules*

$$\underline{\mathsf{PLBA}}([p],[q]) \simeq \bigoplus_{N \geqslant 0} \underline{\mathsf{PLCA}}([p],[N]) \otimes_{\Gamma_N} \underline{\mathsf{PLA}}([N],[q])$$

Proof. The proof is similar to that of Proposition 4.1. The computation can be carried out with the PROPs $\mathsf{PLA}, \mathsf{PLCA}, \mathsf{PLBA}$ introduced in 6.3 since these contain the objects $[p],[N],[q]$. A morphism in PLBA can be represented as an oriented graph obtained from the composition of brackets, cobrackets, permutations, and idempotents. The compatibility (2.14) between δ and μ, and the relations

$$\delta \circ \theta_0 = (\theta_0 \otimes \theta_0) \circ \delta \qquad \theta_0 \circ \mu = \mu \circ (\theta_0 \otimes \theta_0)$$

allow to reorder the morphisms so that the cobrackets precede the brackets, and the idempotent θ_0 occur in between. This yields a surjective map

$$\bigoplus_{N \geqslant 0} (\underline{\mathsf{PLCA}}([p],[N]) \otimes \underline{\mathsf{PLA}}([N],[q])) \to \underline{\mathsf{PLBA}}([p],[q])$$

which factors through the action of $\mathsf{k}[\mathcal{I}^N] \rtimes \mathfrak{S}_N$. The injectivity follows as in 4.1. $\square$

6.5. **Morphisms in $\underline{\mathsf{PLA}}$ and $\underline{\mathsf{PLCA}}$.**

Lemma. *There are isomorphisms of left (resp. right) $\mathsf{k}[\mathcal{I}^N]\rtimes\mathfrak{S}_N$–modules*

$$\underline{\mathsf{PLA}}([N],[p])\simeq\mathsf{k}[\mathcal{I}^N]\otimes\underline{\mathsf{LA}}([N],[p])$$

$$\underline{\mathsf{PLCA}}([p],[N])\simeq\underline{\mathsf{LCA}}([p],[N])\otimes\mathsf{k}[\mathcal{I}^N]$$

where $\mathfrak{S}_N$ acts diagonally on the right–hand side, which are compatible with the action of $\mathfrak{S}_p$.

Proof. We only explain the isomorphism in $\underline{\mathsf{PLA}}$. The result for $\underline{\mathsf{PLCA}}$ follows by observing that $\mathsf{PLCA}\simeq\mathsf{PLA}^{\mathrm{op}}$. Every morphism in $\mathsf{PLA}([N],[p])$ is represented by a linear combination of oriented graphs from N sources to p targets. Since $\mathrm{id}_{[1]}=\theta_0+\theta_1$, all the edges of these graphs can be assumed to be decorated by the idempotents θ_0 or θ_1. The relations

$$\begin{aligned}\theta_0\circ\mu&=\mu\circ(\theta_0\otimes\theta_0)\\ \theta_1\circ\mu&=\mu\circ\big(\theta_1\otimes\theta_1+\theta_1\otimes\theta_0+\theta_0\otimes\theta_1\big)\end{aligned}$$

allow to move all idempotents to the N sources and yield the surjectivity of the map

$$\mathsf{k}[\mathcal{I}^N]\otimes\underline{\mathsf{LA}}([N],[p])\to\underline{\mathsf{PLA}}([N],[p])\qquad\qquad f\otimes P\mapsto f\cdot P$$

Its injectivity follows from the canonical embedding $\underline{\mathsf{LA}}\to\underline{\mathsf{PLA}}$ and the isomorphism

$$\bigoplus_{\underline{i}\in\mathcal{I}^N}\underline{\mathsf{PLA}}([N],[p])\circ\theta_{\underline{i}}\simeq\underline{\mathsf{PLA}}([N],[p])$$

□

6.6. **$\underline{\mathsf{PLBA}}$ and free Lie algebras.** The following is a direct consequence of 6.5, 6.4 and Lemma 4.2.

Proposition.

(1) *There is an isomorphism of $(\mathfrak{S}_q,\mathfrak{S}_p)$–bimodules*

$$\underline{\mathsf{PLBA}}([p],[q])\simeq\bigoplus_{N\geqslant 0}\big((\mathcal{L}_N^{\otimes p})_{\delta_N}\otimes\mathsf{k}[\mathcal{I}^N]\otimes(\mathcal{L}_N^{\otimes q})_{\delta_N}\big)_{\mathfrak{S}_N}$$

(2) *Let $F\in k\mathfrak{S}_p$ and $G\in k\mathfrak{S}_q$ be idempotents, and $F[p]=([p],F)$, $G[q]=([q],G)$ the corresponding objects in $\underline{\mathsf{PLBA}}$. Then one has*

$$\underline{\mathsf{PLBA}}(F[p],G[q])\simeq\bigoplus_{N\geqslant 0}\big(F(\mathcal{L}_N^{\otimes p})_{\delta_N}\otimes\mathsf{k}[\mathcal{I}^N]\otimes G(\mathcal{L}_N^{\otimes q})_{\delta_N}\big)_{\mathfrak{S}_N}$$

In particular,

$$\underline{\mathsf{PLBA}}(\widehat{T}[1]^{\otimes n},T[1]^{\otimes n})\simeq\bigoplus_{N\geqslant 0}\big((T\mathcal{L}_N^{\otimes n})_{\delta_N}\otimes\mathsf{k}[\mathcal{I}^N]\otimes(T\mathcal{L}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N}$$

$$\underline{\mathsf{PLBA}}(\widehat{S}[1]^{\otimes n},S[1]^{\otimes n})\simeq\bigoplus_{N\geqslant 0}\big((S\mathcal{L}_N^{\otimes n})_{\delta_N}\otimes\mathsf{k}[\mathcal{I}^N]\otimes(S\mathcal{L}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N}$$

6.7. **Universal Drinfeld–Yetter modules and $\underline{\mathsf{PLBA}}$.**

Definition. The category $\underline{\mathsf{PDY}}^n$, $n \geqslant 1$, is the colored PROP generated by $n+1$ objects $[1]$ and $\{[\mathsf{V}_k]\}_{k=1,\dots,n}$, and morphisms

$$\mu : [2] \to [1] \qquad \delta : [1] \to [2] \qquad \theta : [1] \to [1]$$

$$\pi_k : [1] \otimes [\mathsf{V}_k] \to [\mathsf{V}_k] \qquad \pi_k^* : [\mathsf{V}_k] \to [1] \otimes [\mathsf{V}_k]$$

such that $([1], \mu, \delta, \theta)$ is a $\underline{\mathsf{PLBA}}$–module in $\underline{\mathsf{PDY}}^n$, and, for every $k = 1, \dots, n$, $([\mathsf{V}_k], \pi_k, \pi_k^*)$ is a Drinfeld–Yetter module over $[1]$.

Set

$$\mathfrak{U}^n_{\mathsf{PDY}} = \mathsf{End}_{\underline{\mathsf{PDY}}^n}\left(\bigotimes_{k=1}^n [\mathsf{V}_k]\right) \tag{6.3}$$

The algebras $\mathfrak{U}^n_{\mathsf{PDY}}$ are universal in the following sense. Let $(\mathfrak{b}, \mathfrak{a})$ be a split pair of bialgebras over k. Then, for any n–tuple $\{V_k, \pi_k, \pi_k^*\}_{k=1}^n$ of Drinfeld–Yetter modules over $\mathfrak{b}$, there is a realisation functor

$$\mathcal{G}_{(\mathfrak{b},\mathfrak{a},V_1,\dots,V_n)} : \underline{\mathsf{PDY}}^n \longrightarrow \mathsf{Vect}_{\mathsf{k}}$$

such that $[\mathfrak{b}] \mapsto \mathfrak{b}$, $[\mathfrak{a}] \mapsto \mathfrak{a}$, and $[\mathsf{V}_k] \mapsto V_k$, $k = 1, \dots, n$.

Proposition. *Let $\mathsf{f} \colon \mathsf{DY}_{\mathfrak{b}} \to \mathsf{Vect}_{\mathsf{k}}$ be the forgetful functor, and $\mathcal{U}^n_{\mathfrak{b}} = \mathsf{End}\left(\mathsf{f}^{\boxtimes n}\right)$. The functors $\mathcal{G}_{(\mathfrak{b},\mathfrak{a},V_1,\dots,V_n)}$ induce an algebra homomorphism*

$$\rho^n_{\mathfrak{b},\mathfrak{a}} : \mathfrak{U}^n_{\mathsf{PDY}} \to \mathcal{U}^n_{\mathfrak{b}}$$

Proof. The proof is identical to that of Proposition 5.3. $\square$

The following is a corollary of the proposition above and 6.1.

Corollary. *Let $\beta, \alpha : \mathfrak{U}^n_{\mathsf{DY}} \to \mathfrak{U}^n_{\mathsf{PDY}}$ be the two algebra homomorphisms defined by the functors $\beta, \alpha : \underline{\mathsf{LBA}} \to \underline{\mathsf{PLBA}}$. For any split pair $(\mathfrak{b}, \mathfrak{a})$, there are commutative diagrams*

$$\begin{tikzcd}
\mathfrak{U}^n_{\mathsf{PDY}} \arrow[r, "\rho^n_{\mathfrak{b},\mathfrak{a}}"] & \mathcal{U}^n_{\mathfrak{b}} \\
\mathfrak{U}^n_{\mathsf{DY}} \arrow[u, "\beta"] \arrow[ur, "\rho^n_{\mathfrak{b}}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathfrak{U}^n_{\mathsf{PDY}} \arrow[r, "\rho^n_{\mathfrak{b},\mathfrak{a}}"] & \mathcal{U}^n_{\mathfrak{b}} \\
\mathfrak{U}^n_{\mathsf{DY}} \arrow[u, "\alpha"] \arrow[r, "\rho^n_{\mathfrak{a}}"'] & \mathcal{U}^n_{\mathfrak{a}} \arrow[u, "\mathrm{Res}^*_{\mathfrak{a}}"']
\end{tikzcd}$$

where $\mathrm{Res}^*_{\mathfrak{a}}$ is the morphism induced by the restriction $\mathsf{DY}_{\mathfrak{b}} \to \mathsf{DY}_{\mathfrak{a}}$, and $\rho^n_{\mathfrak{b}}, \rho^n_{\mathfrak{a}}$ are the homomorphisms defined in 5.3.

6.8. **Universal invariants.** In $\underline{\mathsf{PLBA}}$ we can introduce the notion of invariants with respect to the Lie bialgebra $[\mathfrak{a}]$.

Definition. The subalgebra of $[\mathfrak{a}]$–invariants $(\mathfrak{U}^n_{\mathsf{PDY}})^{[\mathfrak{a}]} \subset \mathfrak{U}^n_{\mathsf{PDY}}$ is the subspace of all $\phi \in \mathfrak{U}^n_{\mathsf{PDY}}$ which commute with the action and the coaction of the Lie bialgebra $[\mathfrak{a}]$ on $[\mathsf{V}_1], \dots, [\mathsf{V}_n]$, that is satisfy

$$\pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \mathrm{id}_{[\mathfrak{b}]} \otimes \phi = \phi \circ \pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \tag{6.4}$$

$$\pi^*_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \phi = \mathrm{id}_{[\mathfrak{b}]} \otimes \phi \circ \pi^*{}_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \tag{6.5}$$

where $\pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} = \pi_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \theta \otimes \mathrm{id}_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]}$ and $\pi^*_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} = \theta \otimes \mathrm{id}_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \pi^*_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]}$.

The following is clear.

Proposition. *The algebra homomorphism $\rho^n_{\mathfrak{b},\mathfrak{a}} : \mathfrak{U}^n_{\mathsf{PDY}} \to \mathcal{U}^n_{\mathfrak{b}}$ restricts to an algebra homomorphism*

$$\rho^n_{\mathfrak{b},\mathfrak{a}} : (\mathfrak{U}^n_{\mathsf{PDY}})^{[\mathfrak{a}]} \to \mathcal{U}^n_{\mathfrak{b},\mathfrak{a}} := \mathsf{End}\left(\mathrm{Res}^{\boxtimes n}_{\mathfrak{a}} : \mathsf{DY}^n_{\mathfrak{b}} \to \mathsf{DY}_{\mathfrak{a}}\right)$$

6.9. **A basis for $\mathfrak{U}^n_{\mathsf{PDY}}$.** The description of the algebras $\mathfrak{U}^n_{\mathsf{PDY}}$ is obtained along the same lines of Propositions 5.2 and 5.12.

Proposition. *The endomorphisms*

$$r^{\underline{i},\sigma}_{\underline{N},\underline{N}'} = \pi^{(\underline{N})} \circ \theta_{\underline{i}} \otimes \mathrm{id} \circ \sigma \otimes \mathrm{id} \circ \pi^{*(\underline{N}')}$$

where $N \geqslant 0$, $\underline{N}, \underline{N}' \in \mathbb{N}^n$ are such that $|\underline{N}| = N = |\underline{N}'|$, $\underline{i} \in \mathcal{I}^N$, and $\sigma \in \mathfrak{S}_N$, are a basis of $\mathfrak{U}^n_{\mathsf{PDY}}$. In particular, the map

$$\xi^n_{\mathsf{PDY}} : \mathfrak{U}^n_{\mathsf{PDY}} \longrightarrow \bigoplus_{N\geqslant 0} \left((FA^{\otimes n}_N)_{\delta_N} \otimes \mathsf{k}[\mathcal{I}^N] \otimes (FA^{\otimes n}_N)_{\delta_N}\right)_{\mathfrak{S}_N}$$

given by $\xi^n_{\mathsf{PDY}}(r^{\underline{i},\sigma}_{\underline{N},\underline{N}'}) = x_{\underline{N}} \otimes \delta_{\underline{i}} \otimes y_{\widetilde{\sigma}(\underline{N}')}$ is a linear isomorphism.

6.10. **PBW theorem for $\mathfrak{U}^n_{\mathsf{PDY}}$.** As in the case of $\mathfrak{U}_{\mathsf{DY}}$, the tower of algebras $\{\mathfrak{U}^n_{\mathsf{PDY}}\}_{n\geqslant 1}$ is endowed with face maps $\Delta^n_i : \mathfrak{U}^n_{\mathsf{PDY}} \to \mathfrak{U}^{n+1}_{\mathsf{PDY}}$ and degeneratation maps $\mathcal{E}^i_n : \mathfrak{U}^n_{\mathsf{PDY}} \to \mathfrak{U}^{n-1}_{\mathsf{PDY}}$ defining a cosimplicial structure.

Let

$$\mathsf{a}^n : \underline{\mathsf{PLBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \to \underline{\mathsf{PDY}}^n(\otimes^n_{k=1}[\mathsf{V}_k], \otimes^n_{k=1}[\mathsf{V}_k])$$

be the map given on $\phi_{\underline{p},\underline{q}} \in \underline{\mathsf{PLBA}}(T^{\underline{p}}[1], T^{\underline{q}}[1])$, by $\mathsf{a}(\phi_{\underline{p},\underline{q}}) = \pi^{(\underline{p})} \circ \phi_{\underline{p},\underline{q}} \circ \pi^{*(\underline{q})}$.

Theorem.

(1) *The following diagram is commutative*

$$\begin{tikzcd}
\underline{\mathsf{PDY}}^n\left(\bigotimes_{k=1}^n[\mathsf{V}_k], \bigotimes_{k=1}^n[\mathsf{V}_k]\right) \arrow[r] & \bigoplus_{N\geqslant 0} \left((\mathcal{A}^{\otimes n}_N)_{\delta_N} \otimes \mathsf{k}[\mathcal{I}^N] \otimes (\mathcal{A}^{\otimes n}_N)_{\delta_N}\right)_{\mathfrak{S}_N} \\
\underline{\mathsf{PLBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \arrow[r] \arrow[u, "\mathsf{a}^n"] & \bigoplus_{N\geqslant 0} \left((T\mathcal{L}^{\otimes n}_N)_{\delta_N} \otimes \mathsf{k}[\mathcal{I}^N] \otimes (T\mathcal{L}^{\otimes n}_N)_{\delta_N}\right)_{\mathfrak{S}_N} \arrow[u] \\
\underline{\mathsf{PLBA}}(\widehat{S}[1]^{\otimes n}, S[1]^{\otimes n}) \arrow[r] \arrow[u, "\mathsf{Sym}"] & \bigoplus_{N\geqslant 0} \left((S\mathcal{L}^{\otimes n}_N)_{\delta_N} \otimes \mathsf{k}[\mathcal{I}^N] \otimes (S\mathcal{L}^{\otimes n}_N)_{\delta_N}\right)_{\mathfrak{S}_N} \arrow[u, "\mathsf{Sym}\otimes\mathrm{id}\otimes\mathsf{Sym}"']
\end{tikzcd}$$

(2) *The map $\mathsf{a}^n \circ \mathsf{Sym}$ is an isomorphism of cosimplicial spaces.*

6.11. **Hochschild cohomology.** The cosimplicial structure on $\{\mathfrak{U}^n_{\mathsf{PDY}}\}_{n\geqslant 1}$ gives rise to the *relative* universal Hochschild complex with differential $d^n = \sum_{i=0}^{n+1}(-1)^i \Delta^n_i :$ $\mathfrak{U}^n_{\mathsf{PDY}} \to \mathfrak{U}^{n+1}_{\mathsf{PDY}}$ The morphisms $\{\rho^n_{\mathfrak{b},\mathfrak{a}}\}_{n\geqslant 1}$ defined in 6.7 define a chain map between the corresponding Hochschild complexes.

Theorem.

(1) *The map* $\mathsf{a}^n \circ \mathsf{Sym}$ *induces an isomorphism*

$$H^i(\mathfrak{U}^\bullet_{\mathsf{PDY}}, d_H) \cong \bigoplus_{j=0}^{i} \underline{\mathsf{PLBA}}\left(\wedge^j[1], \wedge^{i-j}[1]\right)$$

In particular, $H^0(\mathfrak{U}^\bullet_{\mathsf{PDY}}, d_H) = \mathsf{k}$ *and* $H^1(\mathfrak{U}^\bullet_{\mathsf{PDY}}, d_H) = 0$.

(2) *The identification in terms of free Lie algebras of Proposition 6.6 yields*

$$H^i(\mathfrak{U}^\bullet_{\mathsf{PDY}}, d_H) \cong \bigoplus_{N\geqslant 0} \bigoplus_{j=0}^{i} \left[\left(\wedge^j \mathcal{L}_N\right)_{\delta_N} \otimes \mathsf{k}[\mathcal{I}^N] \otimes \left(\wedge^{i-j}\mathcal{L}_N\right)_{\delta_N}\right]_{\mathfrak{S}_N}$$

Proof. The proof is identical to that of Theorem 5.19 where the PROP $\underline{\mathsf{LBA}}$ is replaced by $\underline{\mathsf{PLBA}}$. $\square$

6.12. **Hochschild cohomology and invariants.**

Lemma. $((\mathfrak{U}^n_{\mathsf{PDY}})^{[\mathfrak{a}]}, d^n)$ *is a subcomplex of* $(\mathfrak{U}^n_{\mathsf{PDY}}, d^n)$.

Proof. It is enough to observe that, if $\phi \in \mathfrak{U}^n_{\mathsf{PDY}}$ satisfies (6.4), (6.5), then so does $d^n_i(\phi) \in \mathfrak{U}^{n+1}_{\mathsf{PDY}}$. Namely, let $\mathcal{D}^n_i : \underline{\mathsf{PDY}}^n \to \underline{\mathsf{PDY}}^{n+1}$ be as in 5.7. Then, for any $u \in \mathfrak{U}^n_{\mathsf{PDY}}$, we have

$$\begin{aligned}\pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_{n+1}]} \circ \mathrm{id}_{[\mathfrak{b}]} \otimes d^n_i(u) - d^n_i(u) \circ \pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_{n+1}]}&\\ = \mathcal{D}^n_i\left(\pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \mathrm{id}_{[\mathfrak{b}]} \otimes u - u \circ \pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]}\right)&\end{aligned}$$

Set $\mathcal{D}(\phi) = \sum_i \mathcal{D}^n_i(\phi)$. Then, in particular,

$$\begin{aligned}\pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_{n+1}]} \circ \mathrm{id}_{[\mathfrak{b}]} \otimes d(u) - d(u) \circ \pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_{n+1}]}&\\ = \mathcal{D}\left(\pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \mathrm{id}_{[\mathfrak{b}]} \otimes u - u \circ \pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]}\right)&\end{aligned}$$

$\square$

Let $\wedge^n_{\mathsf{PDY}} \subset \mathfrak{U}^n_{\mathsf{PDY}}$ be the image of the injective map

$$\underline{\mathsf{PLBA}}(\wedge^n[1], \wedge^n[1]) \hookrightarrow \underline{\mathsf{PLBA}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \xrightarrow{\mathsf{a}^n} \mathfrak{U}^n_{\mathsf{PDY}} \tag{6.6}$$

and set $(\wedge^n_{\mathsf{PDY}})^{[\mathfrak{a}]} = \wedge^n_{\mathsf{PDY}} \cap (\mathfrak{U}^n_{\mathsf{PDY}})^{[\mathfrak{a}]}$.

Proposition. $H^n((\mathfrak{U}^\bullet_{\mathsf{PDY}})^{[\mathfrak{a}]}, d^n) \simeq (\wedge^n_{\mathsf{PDY}})^{[\mathfrak{a}]}$.

Proof. Let $f \in \mathfrak{U}^n_{\mathsf{PDY}}$ such that $d(f) = 0$. Then there are unique $d(u) \in \mathfrak{U}^n_{\mathsf{PDY}}$ and $v \in \wedge^n_{\mathsf{PDY}}$ such that $f = v + d(u)$. Namely, let $f = v' + d(u')$ for some $v' \in \wedge^n_{\mathsf{PDY}}$, $u' \in \mathfrak{U}^{n-1}_{\mathsf{PDY}}$. It follows from 6.11 that

$$v - v' = d(u - u') \quad\Longrightarrow\quad v = v' \qquad\text{and}\qquad d(u) = d(u')$$

Assume now $f \in (\mathfrak{U}^n_{\mathsf{PDY}})^{[\mathfrak{a}]}$ and $d(f) = 0$. Since f satisfies (6.4), one has

$$\pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \mathrm{id}_{[\mathfrak{b}]} \otimes v = v \circ \pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \tag{6.7}$$

$$\pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \circ \mathrm{id}_{[\mathfrak{b}]} \otimes d(u) = d(u) \circ \pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]} \tag{6.8}$$

and therefore

$$\pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_{n-1}]} \circ \mathrm{id}_{[\mathfrak{b}]} \otimes u = u \circ \pi_{[\mathfrak{a}],[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_{n-1}]} \tag{6.9}$$

Similarly for (6.5). It follows that $v \in (\wedge^n_{\mathsf{PDY}})^{[\mathfrak{a}]}$ and $u \in (\mathfrak{U}^{n-1}_{\mathsf{PDY}})^{[\mathfrak{a}]}$. $\square$

Corollary. $((\mathfrak{U}^n_{\mathsf{DY}})^{\mathsf{inv}}, d^n)$ *is a subcomplex of* $(\mathfrak{U}^n_{\mathsf{DY}}, d^n)$, *and* $H^n((\mathfrak{U}^\bullet_{\mathsf{DY}})^{\mathsf{inv}}, d^n) \simeq (\wedge^n_{\mathsf{DY}})^{[1]}$.

7. Universal relative twists

In this section, we discuss the existence and uniqueness of invertible elements in the graded completion of $(\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}$ satisfying the relative twist equation (7.1). This implies the uniqueness of the tensor structure on the restriction functor of Drinfeld–Yetter modules corresponding to a split pair of Lie bialgebras.

7.1. **Gradings.** The PROP $\underline{\mathsf{DY}}^n$ has a natural $\mathbb{N}$–bigrading given by $\deg(\sigma)=(0,0)$ for any $\sigma\in\mathfrak{S}_N$,

$$\deg(\mu)=(1,0)=\deg(\pi_{[\mathsf{V}_k]})\qquad\text{and}\qquad\deg(\delta)=(0,1)=\deg(\pi^*_{[\mathsf{V}_k]})$$

for any $1\leq k\leq n$. The algebra $\mathfrak{U}^n_{\mathsf{DY}}$ inherits this bigrading, and $\deg(r^{\sigma}_{\underline{N},\underline{N'}})=(N,N)$, where $r^{\sigma}_{\underline{N},\underline{N'}}$ is the basis element defined in 5.12, and $|\underline{N}|=N=|\underline{N'}|$.

For any $a,b\in\mathbb{N}$, the corresponding $\mathbb{N}$–grading determined by mapping $(1,0),(0,1)$ to a,b respectively yields the same graded completion $\widehat{\mathfrak{U}}^n_{\mathsf{DY}}$ of $\mathfrak{U}^n_{\mathsf{DY}}$, so long as $a+b>0$. For definiteness, we set $a=0$ and $b=1$.

7.2. **Notation.** There is a natural action of $\mathfrak{S}_n$ on $\mathfrak{U}^n_{\mathsf{DY}}$ given by permutations of $[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]$. Specifically, for any $\sigma\in\mathfrak{S}_n$, there is an endofunctor P_σ of $\underline{\mathsf{DY}}^n$ which is the identity on $([1],\mu,\delta)$ and maps each $([V_k],\pi_k,\pi_k^*)$ to $([V_{\sigma(k)}],\pi_{\sigma(k)},\pi^*_{\sigma(k)})$. The action of $\sigma\in\mathfrak{S}_n$ on $\mathfrak{U}^n_{\mathsf{DY}}$ is then defined by $X^\sigma:=\mathsf{Ad}(\sigma)P_\sigma(X)$ for any $X\in\mathfrak{U}^n_{\mathsf{DY}}$. This is a propic version of the action of $\mathfrak{S}_n$ on $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}$.

The generalisation of the insertion/coproduct maps introduced in 5.7 is defined as follows. For any $m\geqslant n$, $1\leqslant i\leqslant m-n+1$, and $X\in\widehat{\mathfrak{U}}^n_{\mathsf{DY}}$, we define $X_{(i,\dots,i+n-1)}\in\widehat{\mathfrak{U}}^m_{\mathsf{DY}}$ by

$$X_{(i,\dots,i+n-1)}:=\mathrm{id}_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_{i-1}]}\otimes X_{[\mathsf{V}_i]\otimes\cdots\otimes[\mathsf{V}_{i+n-1}]}\otimes\mathrm{id}_{[\mathsf{V}_{i+n}]\otimes\cdots\otimes[\mathsf{V}_m]}$$

Then, for any $\sigma\in\mathfrak{S}_m$, we set

$$X_{(\sigma(i),\dots,\sigma(i+n-1))}:=(X_{(i,\dots,i+n-1)})^\sigma$$

For any $p_1,\dots,p_n$ with $p_1+\cdots+p_n=p\leqslant m$, $p_k\neq0$, and $1\leqslant i\leqslant m-p+1$, set $i_k=i+p_1+\cdots+p_{k-1}$, and $I_k=(i_k,\cdots,i_{k+1}-1)$, $k=1,\dots,n$. Then, we define $X_{(I_1,\dots,I_n)}\in\widehat{\mathfrak{U}}^m_{\mathsf{DY}}$ by

$$X_{(I_1,\dots,I_n)}:=\mathrm{id}_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_{i_1-1}]}\otimes X_{[\mathsf{V}_{[i_1,i_2-1]}]\otimes\cdots\otimes[\mathsf{V}_{[i_{n-1},i_n]}]}\otimes\mathrm{id}_{[\mathsf{V}_{i_n+1}]\otimes\cdots\otimes[\mathsf{V}_m]}$$

where $[\mathsf{V}_{[i_1,i_2-1]}]$ denotes the Drinfeld–Yetter module $([\mathsf{V}_{i_1}]\otimes\cdots\otimes[\mathsf{V}_{i_2-1}])$. As before, for any $\sigma\in\mathfrak{S}_m$, we set $X_{\sigma(I_1,\dots,I_n)}=(X_{I_1,\dots,I_n})^\sigma$.

7.3. **Associators.** Define the r–matrix $r=r_{[\mathsf{V}_1],[\mathsf{V}_2]}\in\mathsf{End}_{\underline{\mathsf{DY}}^2}([\mathsf{V}_1]\otimes[\mathsf{V}_2])$ by (2.8), and set $\Omega=r_{12}+r_{21}$.

Definition. An invertible element $\Phi\in\widehat{\mathfrak{U}}^3_{\mathsf{DY}}$ is called an *associator* if the following relations are satisfied (in $\widehat{\mathfrak{U}}^4_{\mathsf{DY}}$ and $\widehat{\mathfrak{U}}^3_{\mathsf{DY}}$ respectively).

- **Pentagon relation**

$$\Phi_{1,2,34}\Phi_{12,3,4}=\Phi_{2,3,4}\Phi_{1,23,4}\Phi_{1,2,3}$$

- **Hexagon relations**

$$e^{\Omega_{12,3}/2}=\Phi_{3,1,2}e^{\Omega_{13}/2}\Phi^{-1}_{1,3,2}e^{\Omega_{23}/2}\Phi_{1,2,3}$$
$$e^{\Omega_{1,23}/2}=\Phi^{-1}_{2,3,1}e^{\Omega_{13}/2}\Phi_{2,1,3}e^{\Omega_{12}/2}\Phi^{-1}_{1,2,3}$$

- **Duality**

$$\Phi_{3,2,1} = \Phi_{1,2,3}^{-1}$$

- 2–**jet**

$$\Phi = 1 + \frac{1}{24}[\Omega_{12}, \Omega_{23}] \qquad \mod (\mathfrak{U}^3_{\mathsf{DY}})_{\geqslant 3}$$

We denote by Assoc the set of associators.

7.4. **Deformation Drinfeld–Yetter modules.** Let $\mathfrak{b}$ be a Lie bialgebra with Drinfeld double $\mathfrak{g}_\mathfrak{b}$. As we explain below, the algebra $\widehat{\mathfrak{U}}^n_{\mathsf{DY}}$ introduced in 7.1 is a universal analogue of the topological algebra $U\mathfrak{g}_\mathfrak{b}^{\otimes n}[\![\hbar]\!]$.

Let for this purpose $\mathsf{DY}^\hbar_\mathfrak{b}$ be the category of Drinfeld–Yetter $\mathfrak{b}$–modules in topologically free $\mathsf{k}[\![\hbar]\!]$–modules. $\mathsf{DY}^\hbar_\mathfrak{b}$ is isomorphic to the category $\mathsf{DY}^{\mathrm{adm}}_{\mathfrak{b}_\hbar}$ of Drinfeld–Yetter modules over the Lie bialgebra $\mathfrak{b}_\hbar = (\mathfrak{b}[\![\hbar]\!], [\cdot,\cdot], \hbar\delta)$, whose coaction is divisible by $\hbar$. We denote by $\widehat{\mathcal{U}}^n_\mathfrak{b}$ the algebra of endomorphisms of the n–fold tensor power of the forgetful functor $\mathsf{f} : \mathsf{DY}^\hbar_\mathfrak{b} \to \mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$. $\widehat{\mathcal{U}}^n_\mathfrak{b}$ identifies canonically with the analogous completion defined for $\mathsf{DY}^{\mathrm{adm}}_{\mathfrak{b}_\hbar}$. Moreoever, the realisation functors

$$\mathcal{G}_{(\mathfrak{b}_\hbar, V_1, \dots, V_n)} : \underline{\mathsf{DY}}^n \longrightarrow \mathrm{Vect}_{\mathsf{k}[\![\hbar]\!]}$$

induce a homomorphism $\widehat{\rho}^n_\mathfrak{b} : \mathfrak{U}^n_{\mathsf{DY}} \to \widehat{\mathcal{U}}^n_\mathfrak{b}$ which naturally extends to $\widehat{\mathfrak{U}}^n_{\mathsf{DY}}$. In particular,[13]

$$\widehat{\rho}^1_\mathfrak{b}(\pi_{[\mathsf{V}_1]} \circ \pi^*_{[\mathsf{V}_1]}) = \hbar \sum_i b_i b^i \qquad \text{and} \qquad \widehat{\rho}^2_\mathfrak{b}(r_{[\mathsf{V}_1],[\mathsf{V}_2]}) = \hbar \sum_i b_i \otimes b^i$$

In Section 14, we shall make use of the following standard construction due to Drinfeld. Let $\Phi \in \widehat{\mathfrak{U}}^3_{\mathsf{DY}}$ be an associator. Then, $\mathsf{DY}^\Phi_\mathfrak{b}$ is the braided monoidal category with the same objects of $\mathsf{DY}^\hbar_\mathfrak{b}$ and commutativity, and associativity constraints given respectively by

$$\beta_\mathfrak{b} = (1\,2) \circ \widehat{\rho}^2_\mathfrak{b}(e^{\Omega/2}) \qquad \text{and} \qquad \Phi_\mathfrak{b} = \widehat{\rho}^3_\mathfrak{b}(\Phi).$$

7.5. **Universal twists in $\underline{\mathsf{DY}}^2$.** The associativity relation (2.9) admits a natural lift to the PROPs $\underline{\mathsf{DY}}^n$.

Proposition. *Let $\Phi \in \mathsf{Assoc}$, and $J \in \widehat{\mathfrak{U}}^2_{\mathsf{DY}}$ be such that*

$$J_{23} \cdot J_{1,23} \cdot \Phi = J_{12} \cdot J_{12,3}$$

Then, for any Lie bialgebra $\mathfrak{b}$, the element $\rho^2_\mathfrak{b}(J) \in \mathcal{U}^2_\mathfrak{b}$ defines a tensor structure on the forgetful functor $\mathsf{f} : \mathsf{DY}^\Phi_\mathfrak{b} \to \mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$.

A simple argument in [1, Section 6.11] shows that the Etingof–Kazhdan tensor structure J^{EK}_{VW} can be lifted to the PROP $\underline{\mathsf{DY}}^2$.

[13] Note that $\mathsf{DY}^\hbar_\mathfrak{b}$ can also be identified with the category of Drinfeld–Yetter modules over the Lie bialgebra $\mathfrak{b}^\hbar = (\mathfrak{b}[\![\hbar]\!], \hbar[\cdot,\cdot], \delta)$ whose action is divisible by $\hbar$. The corresponding realisation functors for $\mathfrak{b}^\hbar$ yield the same homomorphism $\widehat{\rho}^n_\mathfrak{b} : \mathfrak{U}^n_{\mathsf{DY}} \to \widehat{\mathcal{U}}^n_\mathfrak{b}$.

7.6. **Existence of a universal relative twist.** Let $\Phi \in \mathsf{Assoc}$ and let $(\mathfrak{b}, \mathfrak{a})$ be a split pair with corresponding Drinfeld doubles $(\mathfrak{g}_\mathfrak{b}, \mathfrak{g}_\mathfrak{a})$. Let $\Phi_\mathfrak{b}$, $\Phi_\mathfrak{a}$ be the images of Φ in $\widehat{\mathcal{U}}_\mathfrak{b}^3$ and $\widehat{\mathcal{U}}_\mathfrak{a}^3$ respectively.

In [1, Prop. 3.17], we constructed an element $J_\Phi \in \widehat{\mathcal{U}}_\mathfrak{b}^2$, which is invariant under $\mathfrak{a}$, $J_\Phi = 1 \mod \hbar$, and satisfies the *relative twist equation*

$$J_\Phi^{23} J_\Phi^{1,23} \Phi_\mathfrak{b} = \Phi_\mathfrak{a} J_\Phi^{12} J_\Phi^{12,3}$$

We also showed [1, Sec. 7.7] that the construction of J_Φ is universal *i.e.,* that it can be realised as an $[\mathfrak{a}]$–invariant element

$$J_\Phi \in (\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]} \subset \mathsf{End}_{\underline{\mathsf{PDY}}^2}([\mathsf{V}_1] \otimes [\mathsf{V}_2])$$

We summarize this in the following

Theorem. *There is a map* $\mathsf{Assoc} \to (\widehat{\mathfrak{U}}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}$, $\Phi \to J_\Phi$ *such that* $(J_\Phi)_0 = 1$ *and*

$$(\Phi_\beta)_{J_\Phi} = \Phi_\alpha \tag{7.1}$$

where Φ_β, Φ_α *are the images of* Φ *in* $(\widehat{\mathfrak{U}}^3_{\mathsf{PDY}})^{[\mathfrak{a}]}$ *via* α *and* β, *and*

$$\Phi_{J_\Phi} := J_\Phi^{23} J_\Phi^{1,23} \Phi (J_\Phi^{12,3})^{-1} (J_\Phi^{12})^{-1} \tag{7.2}$$

7.7. **Uniqueness of universal relative twists.** We now show the uniqueness of the twist J_Φ up to a unique gauge transformation.

Theorem. *For any* $\Phi \in \mathsf{Assoc}$,

$$\{J \in (\widehat{\mathfrak{U}}^2_{\mathsf{PDY}})^{[\mathfrak{a}]} \mid (\Phi_\beta)_J = \Phi_\alpha, J_0 = 1\} = \{u_1 \cdot u_2 \cdot J_\Phi \cdot u_{12}^{-1} \mid u \in (\widehat{\mathfrak{U}}_{\mathsf{PDY}})^{[\mathfrak{a}]}\}$$

Proof. Assume $J^{(i)} = 1 + \sum_{k \geqslant 1} J_k^{(i)}$, $J_k^{(i)} \in (\mathfrak{U}^2_{\mathsf{PDY}})_k^{[\mathfrak{a}]}$, $i = 1, 2$, and $(\Phi_\beta)_{J^{(i)}} = \Phi_\alpha$. One checks, by linearisation of (7.1), that $J_1^{(i)}$ is an element in $(\mathfrak{U}^2_{\mathsf{PDY}})_1^{[\mathfrak{a}]}$, satisfying $d_H(J_1^{(i)}) = 0$ and

$$\mathsf{Alt}_2(J_1) = \mathsf{Alt}_2(r_\beta - r_\alpha) = \widetilde{r}_\beta - \widetilde{r}_\alpha$$

Up to a gauge, we may assume $J_1^{(i)} = \widetilde{r}_\beta - \widetilde{r}_\alpha$, $i = 1, 2$. We want to show that there exists an invertible $u \in (\widehat{\mathfrak{U}}_{\mathsf{PDY}})^{[\mathfrak{a}]}$ such that

$$u_1 \cdot u_2 \cdot J^{(1)} \cdot u_{12}^{-1} = J^{(2)} \tag{7.3}$$

Assume that (7.3) is true modulo $(\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}_{\geqslant n}$, *i.e.,* there exists an invertible element $u^{(n-1)} \in (\widehat{\mathfrak{U}}_{\mathsf{PDY}})^{[\mathfrak{a}]}$ such that

$$u_1^{(n-1)} \cdot u_2^{(n-1)} \cdot J^{(1)} \cdot {u_{12}^{(n-1)}}^{-1} = J^{(2)} \mod (\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}_{\geqslant n} \tag{7.4}$$

Let now $\widetilde{J}^{(1)}$ be the left–hand side of (7.4), and $\eta \in (\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}_n$ such that

$$J^{(2)} = \widetilde{J}^{(1)} + \eta \qquad \mod (\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}_{\geqslant n+1}$$

One checks that $\widetilde{J}^{(1)}$ satisfies $(\Phi_\beta)_{\widetilde{J}^{(1)}} = \Phi_\alpha$ modulo $(\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}_{\geqslant n+1}$. Comparing with the equation $(\Phi_\beta)_{J^{(2)}} = \Phi_\alpha$ modulo $(\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}_{\geqslant n+1}$, one gets

$$\eta_{23} + (\eta)_{1,23} - (\eta)_{12,3} - \eta_{12} = 0$$

that is $d_H(\eta) = 0$. Therefore, by Proposition 6.12, there exist a unique $v \in (\mathfrak{U}_{\mathsf{PDY}})^{[\mathfrak{a}]}_n$ and $\mu \in (\wedge^2_{\mathsf{PDY}})^{[\mathfrak{a}]}_n$, such that $\eta = d_H(v) + \mu$.

We claim $\mu = 0$. Then, we may set $u^{(n)} = (1-v)u^{(n-1)}$, and we get

$$u_1^{(n)} \cdot u_2^{(n)} \cdot J^{(1)} \cdot u_{12}^{(n)\,-1} = J^{(2)} \qquad \text{mod } (\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}_{\geqslant n+1}$$

There remains to prove the claim. Set $\widetilde{J}^{(2)} = {u^{(n)}}_1 \cdot {u^{(n)}}_2 \cdot J^{(1)} \cdot {u^{(n)}}_{12}^{-1}$. Then

$$\widetilde{J}^{(2)} = J^{(2)} + \mu \qquad \text{mod } (\mathfrak{U}^2_{\mathsf{PDY}})^{[\mathfrak{a}]}_{\geqslant n+1}$$

Let $J^{(2)}_{[n+1]}, \widetilde{J}^{(2)}_{[n+1]}$ be the corresponding truncations. We set

$$\xi = (J^{(2)}_{[n+1]})^{23}(J^{(2)}_{[n+1]})^{1,23}\Phi_\beta - \Phi_\alpha(J^{(2)}_{[n+1]})^{12}(J^{(2)}_{[n+1]})^{12,3} \qquad \text{mod } (\mathfrak{U}^3_{\mathsf{PDY}})^{[\mathfrak{a}]}_{\geqslant n+2}$$
$$\widetilde{\xi} = (\widetilde{J}^{(2)}_{[n+1]})^{23}(\widetilde{J}^{(2)}_{[n+1]})^{1,23}\Phi_\beta - \Phi_\alpha(\widetilde{J}^{(2)}_{[n+1]})^{12}(\widetilde{J}^{(2)}_{[n+1]})^{12,3} \qquad \text{mod } (\mathfrak{U}^3_{\mathsf{PDY}})^{[\mathfrak{a}]}_{\geqslant n+2}$$

Since $\widetilde{J}^{(2)}$ and $J^{(2)}$ are both solutions of $(\Phi_\beta)_J = \Phi_\alpha$, it follows

$$\xi = d_H\left(J^{(2)}_{n+1}\right) \qquad \text{and} \qquad \widetilde{\xi} = d_H\left(\widetilde{J}^{(2)}_{n+1}\right)$$

Therefore $d_H\xi = d_H\widetilde{\xi} = 0$ and $\mathsf{Alt}\xi = \mathsf{Alt}\widetilde{\xi} = 0$. We then observe that

$$\widetilde{\xi} - \xi = f(\mu)$$

where $f(\mu) = A_r^{23}(\mu^{12}+\mu^{13}) + \mu^{23}(A_r^{12}+A_r^{13}) - A_r^{12}(\mu^{13}+\mu^{23}) - \mu^{12}(A_r^{13}+A_r^{23})$ and $A_r = \widetilde{r}_\beta - \widetilde{r}_\alpha$. By straightforward computation, one checks

$$\mathsf{Alt}f(\mu) = [\![\widetilde{r}_\beta - \widetilde{r}_\alpha, \mu]\!]$$

where $[\![,]\!]$ is the Schouten bracket from $\wedge^2_{\mathsf{PDY}} \to \wedge^3_{\mathsf{PDY}}$. Therefore $[\![\widetilde{r}_\beta - \widetilde{r}_\alpha, \mu]\!] = 0$. Since $[\![\widetilde{r}_\beta - \widetilde{r}_\alpha, -]\!] = [\![\widetilde{r}_\beta, -]\!]$ on $(\wedge^2_{\mathsf{PDY}})^{[\mathfrak{a}]}$, one gets $[\![\widetilde{r}_\beta, \mu]\!] = 0$. It follows from [12, Prop. 2.2] that the map $[\![\widetilde{r}_\beta, -]\!]$ has a trivial kernel on $\wedge^2_{\mathsf{DY}}$ and $\wedge^2_{\mathsf{PDY}}$. Therefore $\mu = 0$, and the theorem is proved. □

Remark. Theorem 7.7 generalises [12, Thm. 2.1], where it is proved for the PROP $\underline{\mathsf{LBA}}$. In particular, the uniqueness of the twist in $\underline{\mathsf{LBA}}$ can be recovered by applying the forgetful functor $\underline{\mathsf{PLBA}} \to \underline{\mathsf{LBA}}$. Theoren 7.7 also generalises [26, Thm 6.1], where it is proved for a semisimple Lie algebra.

8. Lie bialgebras graded by semigroups

In this section, we review some basic facts about partial semigroups and Lie (co–)algebras graded by these.

8.1. **Partial semigroups.** A *partial semigroup* is a pair (S, σ), where S is a set, and $\sigma : \mathsf{S} \times \mathsf{S} \to \mathsf{S}$ a partial map defined on a subset $\mathsf{S}^{(2)} \subseteq \mathsf{S} \times \mathsf{S}$ such that, for any $\alpha, \beta, \gamma \in \mathsf{S}$,

$$\sigma(\sigma(\alpha,\beta),\gamma) = \sigma(\alpha, \sigma(\beta,\gamma))$$

when both sides are defined, that is if (α,β), $(\sigma(\alpha,\beta),\gamma)$, (β,γ), $(\alpha,\sigma(\beta,\gamma)) \in \mathsf{S}^{(2)}$.

Remark. It is common in the literature (see *e.g.*, [16]) to assume that the semigroup law σ is strongly associative, *i.e.,* for any $\alpha, \beta, \gamma \in \mathsf{S}$,

$$(\alpha,\beta), (\sigma(\alpha,\beta),\gamma) \in \mathsf{S}^{(2)} \quad \text{if and only if} \quad (\beta,\gamma), (\alpha,\sigma(\beta,\gamma)) \in \mathsf{S}^{(2)}.$$

This definition is stronger than the one given above, and is not suited for our purposes, since it does not hold for root systems (cf. 12.6).[14]

8.2. **Coherence.** Every partial semigroup satisfies the following coherence property. Let $\mathsf{Br}(n)$ be the set of full bracketings on the non–associative monomial $x_1 \cdots x_n$. Let $\sigma_b : \mathsf{S}^n \to \mathsf{S}$ be the partial map obtained by composing σ along b (*e.g.*, $\sigma_{(\bullet\bullet)\bullet}(\alpha,\beta,\gamma) = \sigma(\sigma(\alpha,\beta),\gamma)$). Set

$$\mathsf{S}^{(n)} = \{\underline{\alpha} \in \mathsf{S}^n \mid \sigma_b(\alpha) \text{ is defined for any } b \in \mathsf{Br}(n)\}$$

Proposition. *For any $\underline{\alpha} \in \mathsf{S}^{(n)}$, and $b, b' \in \mathsf{Br}(n)$, $\sigma_b(\underline{\alpha}) = \sigma_{b'}(\underline{\alpha})$.*

Proof. Let $b, b' \in \mathsf{Br}(n)$ two bracketings which differ by an elementary move, *i.e.,* there are $i < j < k < l$ such that, up to a permutation $b \leftrightarrow b'$

$$b \ = \cdots \big(\big((x_{i+1} \cdots x_j)(x_{j+1} \cdots x_k)\big)(x_{k+1} \cdots x_l)\big) \cdots$$
$$b' = \cdots \big((x_{i+1} \cdots x_j)\big((x_{j+1} \cdots x_k)(x_{k+1} \cdots x_l)\big)\big) \cdots$$

and they agree on everything else. Let $\underline{\alpha} \in \mathsf{S}^{(n)}$, and set $\alpha = \sigma_{b_{ij}}(\alpha_{i+1}, \ldots, \alpha_j)$, $\beta = \sigma_{b_{jk}}(\alpha_{j+1}, \ldots, \alpha_k)$, and $\gamma = \sigma_{b_{kl}}(\alpha_{k+1}, \ldots, \alpha_l)$, where b_{rs} is the restriction of b and b' to $(x_{r+1} \cdots x_s)$. By associativity, $\sigma(\sigma(\alpha,\beta),\gamma) = \sigma(\alpha, \sigma(\beta,\gamma))$ so that $\sigma_b(\underline{\alpha}) = \sigma_{b'}(\underline{\alpha})$. Since for any $b, b' \in \mathsf{Br}(n)$, there is a sequence $b = b_0, b_1, \ldots, b_r = b'$ such that b_i, b_{i+1} differ by an elementary move, $\sigma_b(\underline{\alpha}) = \sigma_{b'}(\underline{\alpha})$. $\square$

8.3. **Morphisms, subsemigroups and saturated subsets.** Let S, T be partial semigroups. A *morphism* $\phi : \mathsf{S} \to \mathsf{T}$ is a map such that $(\alpha,\beta) \in \mathsf{S}^{(2)}$ if and only if $(\phi(\alpha), \phi(\beta)) \in \mathsf{T}^{(2)}$, and $\phi(\sigma_{\mathsf{S}}(\alpha,\beta)) = \sigma_{\mathsf{T}}(\phi(\alpha),\phi(\beta))$ for any $(\alpha,\beta) \in \mathsf{S}^{(2)}$.

Any subset $\mathsf{S}' \subseteq \mathsf{S}$ inherits a partial semigroup structure. Namely, we denote by $\mathsf{t}(\mathsf{S}')$ the semigroup with underlying set S',

$$\mathsf{t}(\mathsf{S}')^{(2)} = \{(\alpha,\beta) \in \mathsf{S}' \times \mathsf{S}' \,|\, (\alpha,\beta) \in \mathsf{S}^{(2)} \text{ and } \sigma(\alpha,\beta) \in \mathsf{S}'\}$$

and semigroup law induced by that of S. The corresponding embedding $\mathsf{t}(\mathsf{S}') \to \mathsf{S}$ is a morphism of semigroups if and only if S' is a *subsemigroup* of S *i.e.,* if $(\alpha,\beta) \in (\mathsf{S}' \times \mathsf{S}') \cap \mathsf{S}^{(2)}$ implies $\sigma(\alpha,\beta) \in \mathsf{S}'$.

For any $\alpha \in \mathsf{S}$, set

$$\mathsf{S}^{(2)}_{\alpha} = \{(\beta,\gamma) \in \mathsf{S}^{(2)} \mid \sigma(\beta,\gamma) = \alpha\}$$

A subset $\mathsf{S}' \subseteq \mathsf{S}$ is *saturated* if $\mathsf{S}^{(2)}_{\alpha} \subseteq \mathsf{S}' \times \mathsf{S}'$ for any $\alpha \in \mathsf{S}'$.

A partial semigroup is *commutative* if $\mathsf{S}^{(2)}$ is symmetric, *i.e.,* $(\alpha,\beta) \in \mathsf{S}^{(2)}$ if and only if $(\beta,\alpha) \in \mathsf{S}^{(2)}$, in which case $\sigma(\alpha,\beta) = \sigma(\beta,\alpha)$.

Henceforth, by semigroup we mean a commutative partial semigroup $(\mathsf{S}, +)$.

8.4. **S–graded Lie (co)algebras.** Let S be a semigroup, and $\mathcal{N}$ a k–linear symmetric monoidal category $\mathcal{N}$.

Definition.

(1) An object $\mathfrak{b}$ in $\mathcal{N}$ is S–*graded* if it decomposes as $\mathfrak{b} = \bigoplus_{\alpha \in \mathsf{S}} \mathfrak{b}_{\alpha}$.
(2) A morphism $\phi : \mathfrak{b}' \to \mathfrak{b}$ between S–graded objects in $\mathcal{N}$ is *homogeneous* if $\phi(\mathfrak{b}'_{\alpha}) \subseteq \mathfrak{b}_{\alpha}$ for any $\alpha \in \mathsf{S}$.

[14] For example, in the root system of $\mathfrak{sl}_4$, $\alpha_2 + \alpha_1$ and $(\alpha_2 + \alpha_1) + \alpha_3$ are defined, but $\alpha_1 + \alpha_3$ is not.

If $\mathfrak{b} \in \mathcal{N}$ is S–graded, then $\mathfrak{b} \otimes \mathfrak{b}$ is $\mathsf{S} \times \mathsf{S}$–graded, and the subspace

$$\mathfrak{b}_{\mathsf{S}^{(2)}} = \bigoplus_{(\beta,\gamma)\in\mathsf{S}^{(2)}} \mathfrak{b}_\beta \otimes \mathfrak{b}_\gamma \subseteq \mathfrak{b} \otimes \mathfrak{b}$$

is naturally S–graded with $(\mathfrak{b}_{\mathsf{S}^{(2)}})_\alpha = \bigoplus_{(\beta,\gamma)\in\mathsf{S}^{(2)}_\alpha} \mathfrak{b}_\beta \otimes \mathfrak{b}_\gamma$. Let $i_{\mathsf{S}^{(2)}} : \mathfrak{b}_{\mathsf{S}^{(2)}} \to \mathfrak{b} \otimes \mathfrak{b}$ and $p_{\mathsf{S}^{(2)}} : \mathfrak{b} \otimes \mathfrak{b} \to \mathfrak{b}_{\mathsf{S}^{(2)}}$ be the canonical injection and projection respectively, $\theta_{\mathsf{S}^{(2)}} = i_{\mathsf{S}^{(2)}} \circ p_{\mathsf{S}^{(2)}} : \mathfrak{b} \otimes \mathfrak{b} \to \mathfrak{b} \otimes \mathfrak{b}$ the corresponding idempotent, and set $\overline{\theta}_{\mathsf{S}^{(2)}} = \mathrm{id}_{\mathfrak{b}\otimes\mathfrak{b}} - \theta_{\mathsf{S}^{(2)}}$.

Definition.

(1) A Lie algebra $(\mathfrak{b}, [\,,\,])$ in $\mathcal{N}$ is S–*graded* if $\mathfrak{b}$ is S–graded, $[\,,\,] \circ \overline{\theta}_{\mathsf{S}^{(2)}} = 0$ and $[\,,\,] \circ i_{\mathsf{S}^{(2)}} : \mathfrak{b}_{\mathsf{S}^{(2)}} \to \mathfrak{b}$ is homogeneous.
(2) A Lie coalgebra $(\mathfrak{b}, \delta)$ in $\mathcal{N}$ is S–*graded* if $\mathfrak{b}$ is S–graded, $\overline{\theta}_{\mathsf{S}^{(2)}} \circ \delta = 0$ and $p_{\mathsf{S}^{(2)}} \circ \delta : \mathfrak{b} \to \mathfrak{b}_{\mathsf{S}^{(2)}}$ is homogeneous.

8.5. Let $\mathfrak{b} \in \mathcal{N}$ be an S–graded object. For any subset $\mathsf{S}' \subseteq \mathsf{S}$, set $\mathfrak{b}' = \bigoplus_{\alpha\in\mathsf{S}'} \mathfrak{b}_\alpha$ and let $i : \mathfrak{b}' \to \mathfrak{b}$ and $p : \mathfrak{b} \to \mathfrak{b}'$ be the corresponding injection and projection. The following is straightforward.

Proposition.

(1) *Assume $\mathfrak{b}$ is an S–graded Lie algebra and set $\mu' = p \circ \mu \circ i \otimes i$.*
  (a) *If S' is a subsemigroup of S, then $(\mathfrak{b}', \mu')$ is an S'–graded Lie algebra, and $i : \mathfrak{b}' \to \mathfrak{b}$ is a morphism of Lie algebras.*
  (b) *If S' is a saturated subset of S, then $(\mathfrak{b}', \mu')$ is a $\mathsf{t}(\mathsf{S}')$–graded Lie algebra, and $p : \mathfrak{b} \to \mathfrak{b}'$ is a morphism of Lie algebras.*
(2) *Assume $\mathfrak{b}$ is an S–graded Lie coalgebra, and set $\delta' = p \otimes p \circ \delta \circ i$.*
  (a) *If S' is a subsemigroup of S, then $(\mathfrak{b}', \delta')$ is an S'–graded Lie coalgebra, and $p : \mathfrak{b} \to \mathfrak{b}'$ is a morphism of Lie coalgebras.*
  (b) *If S' is a saturated subset of S, then $(\mathfrak{b}', \delta')$ is a $\mathsf{t}(\mathsf{S}')$–graded Lie coalgebra, and $i : \mathfrak{b}' \to \mathfrak{b}$ is a morphism of Lie coalgebras.*
(3) *In particular, if $(\mathfrak{b}, \mu, \delta)$ is an S–graded Lie bialgebra and $\mathsf{S}' \subseteq \mathsf{S}$ a saturated subsemigroup, then $(\mathfrak{b}', \mu', \delta')$ is an S'–graded Lie bialgebra, and $(\mathfrak{b}, \mathfrak{b}')$ is a split pair of Lie bialgebras with respect to i, p.*

8.6. **Example.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra with fixed Borel and Cartan subalgebras $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ and standard Lie bialgebra structure (see §15.5), and let $\mathsf{R}_+ \subset \mathfrak{h}^*$ be the semigroup of positive roots of $\mathfrak{g}$ relative to $\mathfrak{b}$. Let R_0 be the semigroup with underlying set $\mathsf{R}_+ \sqcup \{0\}$, and law extending that of R_+ by an element 0 such that $\alpha + 0 = \alpha$ for any $\alpha \in \mathsf{R}_+$, with $0 + 0$ not defined. Then, $\mathfrak{b}$ is graded as a Lie bialgebra by R_0, with $\mathfrak{b}_0 = \mathfrak{h}$, and $\mathfrak{b}_\alpha = \mathfrak{g}_\alpha$, $\alpha \in \mathsf{R}_+$.

Let D be the Dynkin diagram of $\mathfrak{g}$, $B \subseteq D$ a subdiagram, and $\mathsf{R}_{B,+} \subseteq \mathsf{R}_+$ the subset of roots whose support lies in B. $\mathsf{R}_{B,+} \sqcup \{0\}$ is a saturated subsemigroup of R_0, and the corresponding Lie subbialgebra of $\mathfrak{b}$ is $\mathfrak{b}_B = \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathsf{R}_{B,+}} \mathfrak{g}_\alpha$.

9. Semigroup extensions of $\underline{\mathsf{LBA}}$

In this section, we introduce the PROP $\underline{\mathsf{LBA}}_{\mathsf{S}}$ which governs Lie bialgebras graded by a given semigroup S.

9.1. We construct below an S–graded version of the PROPs $\underline{\mathsf{LA}}, \underline{\mathsf{LCA}}, \underline{\mathsf{LBA}}$ by adding a complete family of orthogonal idempotents labeled by the elements of the semigroup S. We describe in details the refinement of $\underline{\mathsf{LA}}$, which is easily adapted to $\underline{\mathsf{LCA}}$ and $\underline{\mathsf{LBA}}$.

9.1.1. Let $\widetilde{\mathsf{LA}}_{\mathsf{S}}$ be the PROP generated by morphisms $\mu : [2] \to [1]$ and $\theta_\alpha : [1] \to [1]$, $\alpha \in \mathsf{S}$, with relations (2.10),

$$\theta_\alpha \circ \theta_\beta = \delta_{\alpha,\beta} \cdot \theta_\alpha \tag{9.1}$$

for any $\alpha, \beta \in \mathsf{S}$, and

$$\mu \circ \theta_\beta \otimes \theta_\gamma = \begin{cases} \theta_{\beta+\gamma} \circ \mu \circ \theta_\beta \otimes \theta_\gamma & \text{if} \quad (\beta,\gamma) \in \mathsf{S}^{(2)} \\ 0 & \text{if} \quad (\beta,\gamma) \notin \mathsf{S}^{(2)} \end{cases} \tag{9.2}$$

9.1.2. In addition to the orthogonality condition (9.1), we wish to impose the completeness relation

$$\sum_{\alpha \in \mathsf{S}} \theta_\alpha = \mathrm{id}_{[1]} \tag{9.3}$$

and, more generally, $\sum_{\underline{\alpha} \in \mathsf{S}^p} \theta_{\underline{\alpha}} = \mathrm{id}_{[p]}$ for any $p \in \mathbb{N}$, where

$$\theta_{\underline{\alpha}} = \theta_{\alpha_1} \otimes \cdots \otimes \theta_{\alpha_p} \in \mathrm{End}([p]) \qquad \underline{\alpha} = (\alpha_1, \ldots, \alpha_p) \in \mathsf{S}^p$$

To this end, let $\mathsf{k}[\mathsf{S}^p]^{\mathsf{fin}}$ be the functions on S^p with finite support, regarded as a non–unital algebra irrespective of whether S is finite. Then, $\widetilde{\mathsf{LA}}_{\mathsf{S}}([p],[q])$ is a $(\mathsf{k}[\mathsf{S}^q]^{\mathsf{fin}}, \mathsf{k}[\mathsf{S}^p]^{\mathsf{fin}})$–bimodule, with the functions $\delta_{\underline{\alpha}}$, $\alpha \in \mathsf{S}^q$, and $\delta_{\underline{\beta}}$, $\beta \in \mathsf{S}^p$, acting as $\theta_{\underline{\alpha}} \circ -$ and $- \circ \theta_{\underline{\beta}}$ respectively. We denote by LA_{S} the PROP with morphisms

$$\mathsf{LA}_{\mathsf{S}}([p],[q]) \quad = \quad \mathsf{k}[\mathsf{S}^q]^{\mathsf{fin}} \otimes_{\mathsf{k}[\mathsf{S}^q]^{\mathsf{fin}}} \widetilde{\mathsf{LA}}_{\mathsf{S}}([p],[q]) \hat{\otimes}_{\mathsf{k}[\mathsf{S}^p]^{\mathsf{fin}}} \mathsf{k}[\mathsf{S}^p]$$

where the tensor product is completed with respect to the weak topology on $\mathsf{k}[\mathsf{S}^p]$.[15] Explicitly, one has

$$\mathsf{LA}_{\mathsf{S}}([p],[q]) \quad = \quad \prod_{\underline{\alpha} \in \mathsf{S}^p} \bigoplus_{\underline{\beta} \in \mathsf{S}^q} \theta_{\underline{\beta}} \circ \widetilde{\mathsf{LA}}_{\mathsf{S}}([p],[q]) \circ \theta_{\underline{\alpha}}$$

The composition of morphisms $\mathsf{LA}_{\mathsf{S}}([p],[q]) \otimes \mathsf{LA}_{\mathsf{S}}([q],[r]) \to \mathsf{LA}_{\mathsf{S}}([p],[r])$ in LA_{S} is induced by that in $\widetilde{\mathsf{LA}}_{\mathsf{S}}$, because the multiplication of $f \in \mathsf{k}[\mathsf{S}^q]$ and $g \in \mathsf{k}[\mathsf{S}^q]^{\mathsf{fin}}$ has finite support. The identity on $[p]$ in LA_{S} is precisely the element $\sum_{\underline{\alpha} \in \mathsf{S}^p} \theta_{\underline{\alpha}}$.

The following is straightforward.

Lemma. *In LA_{S}, the compatibility condition (9.2) is equivalent to*

$$\theta_\alpha \circ \mu = \sum_{(\beta',\gamma') \in \mathsf{S}^{(2)}_\alpha} \mu \circ \theta_{\beta'} \otimes \theta_{\gamma'}$$

where $\mathsf{S}^{(2)}_\alpha = \{(\beta',\gamma') \in \mathsf{S}^{(2)} \mid \beta' + \gamma' = \alpha\}$.

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[15] A basis of neighborhoods of zero in $\mathsf{k}[\mathsf{S}^p]$ is given by the subsets

$$U_{\underline{\alpha}_1,\ldots,\underline{\alpha}_r} := \{f \in \mathsf{k}[\mathsf{S}^p] \mid \forall i = 1,\ldots,r, \;\; f(\underline{\alpha}_i) = 0\}$$

Then $\mathsf{k}[\mathsf{S}^p] = \lim_U \mathsf{k}[\mathsf{S}^p]/U$ where the limit runs over the open subsets $U \subseteq \mathsf{k}[\mathsf{S}^p]$, and, for any discrete $\mathsf{k}[\mathsf{S}^p]^{\mathsf{fin}}$–module V,

$$\mathsf{k}[\mathsf{S}^p] \hat{\otimes}_{\mathsf{k}[\mathsf{S}^p]^{\mathsf{fin}}} V = \lim_U \left(\mathsf{k}[\mathsf{S}^p]/U \otimes_{\mathsf{k}[\mathsf{S}^p]^{\mathsf{fin}}} V \right)$$

Finally, we denote by $\underline{\mathsf{LA}}_{\mathsf{S}}$ the closure under infinite direct sums of the Karoubi envelope of LA_{S}.

9.1.3. The PROPs $\underline{\mathsf{LCA}}_{\mathsf{S}}$ and $\underline{\mathsf{LBA}}_{\mathsf{S}}$ are obtained similarly. In particular, we impose the following compatibility condition between the idempotents $\{\theta_\alpha\}_{\alpha\in\mathsf{S}}$ and the cobracket $\delta:[2]\to[1]$,

$$\theta_\beta\otimes\theta_\gamma\circ\delta=\begin{cases}\theta_\beta\otimes\theta_\gamma\circ\delta\circ\theta_{\beta+\gamma} & \text{if}\quad(\beta,\gamma)\in\mathsf{S}^{(2)}\\ 0 & \text{if}\quad(\beta,\gamma)\notin\mathsf{S}^{(2)}\end{cases}\tag{9.4}$$

Analogously to Proposition 9.1, in $\underline{\mathsf{LCA}}_{\mathsf{S}}$ and $\underline{\mathsf{LBA}}_{\mathsf{S}}$ this is equivalent to the condition

$$\delta\circ\theta_\alpha=\sum_{(\beta,\gamma)\in\mathsf{S}^{(2)}_\alpha}\theta_\beta\otimes\theta_\gamma\circ\delta$$

9.1.4. We observe that, although not strictly necessary, the commutativity of S is a natural requirement in the case of PROPs describing Lie operations. Namely, one has, for any $\alpha,\beta,\gamma\in\mathsf{S}$,

$$\begin{aligned}\delta_{\alpha,\beta+\gamma}\cdot\mu\circ\theta_\beta\otimes\theta_\gamma=&\ \theta_\alpha\circ\mu\circ\theta_\beta\otimes\theta_\gamma=-\theta_\alpha\circ\mu\circ\theta_\gamma\otimes\theta_\beta\circ(1\,2)\\ &=-\delta_{\alpha,\gamma+\beta}\cdot\mu\circ\theta_\gamma\otimes\theta_\beta\circ(1\,2)=\delta_{\alpha,\gamma+\beta}\cdot\mu\circ\theta_\beta\otimes\theta_\gamma\end{aligned}$$

and $\mu\circ\theta_\beta\otimes\theta_\gamma=\delta_{\beta+\gamma,\gamma+\beta}\cdot\mu\circ\theta_\beta\otimes\theta_\gamma$. In particular, we see that $\mu\circ\theta_\beta\otimes\theta_\gamma=0$ if either (β,γ) or (γ,β) are not in $\mathsf{S}^{(2)}$, or $\beta+\gamma\neq\gamma+\beta$.

Remark. Let $\mathcal{N}$ be a Karoubian, k–linear symmetric monoidal category. There is a canonical isomorphism

$$\mathsf{Fun}^{\otimes}_{\mathsf{b}}(\underline{\mathsf{LBA}}_{\mathsf{S}},\mathcal{N})\simeq\mathsf{LBA}_{\mathsf{S}}(\mathcal{N})$$

where $\mathsf{LBA}_{\mathsf{S}}(\mathcal{N})$ is the category of S–graded Lie bialgebras in $\mathcal{N}$.

9.2. **Examples.**

(1) For $\mathsf{S}=\{0\}$, one has $\theta_0=\mathrm{id}$ and $\underline{\mathsf{LBA}}_{\mathsf{S}}=\underline{\mathsf{LBA}}$.
(2) Let $\mathsf{S}=\{0,1\}$ be the semigroup with the addition table

+	0	1
0	0	1
1	1	1

θ_0 is a morphism of Lie bialgebras, $\theta_1=\mathrm{id}_{[1]}-\theta_0$, and it is immediate to check that $\underline{\mathsf{LBA}}_{\mathsf{S}}=\underline{\mathsf{PLBA}}$ as described in 6.3.[16]
(3) More generally, if $\mathsf{S}=\{0,\dots,n\}$ with the tropical addition law $p+q=\max(p,q)$, a module over $\underline{\mathsf{LBA}}_{\mathsf{S}}$ consists of a Lie bialgebra $\mathfrak{b}$ endowed with a sequence $\mathfrak{b}_0\hookrightarrow\cdots\hookrightarrow\mathfrak{b}_n=\mathfrak{b}$ of split inclusions of Lie bialgebras. The Lie subbialgebra $\mathfrak{b}_i$ is the direct sum $\bigoplus_{p=0}^i\mathrm{Im}(\theta_p)$.

[16]Note that the equality $\underline{\mathsf{LBA}}_{\mathsf{S}}=\underline{\mathsf{PLBA}}$ holds only after taking Karoubi envelopes. Indeed, $\mathsf{LBA}_{\mathsf{S}}$ is generated by one object, while PLBA is generated by the objects $[\mathfrak{a}],[\mathfrak{b}]$.

9.3. **Morphisms in $\underline{\mathsf{LBA}}_{\mathsf{S}}$.** The projections

$$\theta_{\underline{\alpha}} = \theta_{\alpha_1} \otimes \cdots \otimes \theta_{\alpha_N}, \quad \underline{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathsf{S}^N$$

are a complete family of orthogonal idempotents in $\underline{\mathsf{LA}}_{\mathsf{S}}([N],[N])$ and $\underline{\mathsf{LCA}}_{\mathsf{S}}([N],[N])$. By construction, there is a natural right (resp. left) action of $\Gamma_{\mathsf{S},N} = \mathfrak{S}_N \ltimes \mathsf{k}[\mathsf{S}^N]$ on $\underline{\mathsf{LA}}_{\mathsf{S}}([N],[q])$ and $\underline{\mathsf{LCA}}_{\mathsf{S}}([p],[N])$, and natural identifications

$$\underline{\mathsf{LA}}_{\mathsf{S}}([N],[q]) \simeq \prod_{\underline{\alpha} \in \mathsf{S}^N} \underline{\mathsf{LA}}([N],[q]) \circ \theta_{\underline{\alpha}} \tag{9.5}$$

and

$$\underline{\mathsf{LCA}}_{\mathsf{S}}([p],[N]) \simeq \prod_{\underline{\alpha} \in \mathsf{S}^N} \theta_{\underline{\alpha}} \circ \underline{\mathsf{LCA}}([p],[N]) \tag{9.6}$$

The description of the morphisms in $\underline{\mathsf{LBA}}_{\mathsf{S}}$ is similar to those in $\underline{\mathsf{LBA}}$ and $\underline{\mathsf{PLBA}}$ (cf. 6.6), but the commutativity relations (9.2) and (9.4),

$$\forall\, (\beta, \gamma) \notin \mathsf{S}^{(2)}, \qquad \mu \circ \theta_\beta \otimes \theta_\gamma = 0 = \theta_\beta \otimes \theta_\gamma \circ \delta$$

require the replacement of the free Lie algebras $\mathcal{L}_N$ with the Lie algebras $\mathcal{L}_{N,\underline{\alpha}}$, $\underline{\alpha} \in \mathsf{S}^N$ defined as follows. As we explained in 4.1, it is convenient to describe the Lie algebras $\mathcal{L}_{N,\underline{\alpha}}$ in terms of labeled binary trees. Set $X_{\underline{\alpha}} = \{\alpha_1, \ldots, \alpha_N\}$ and let $\mathsf{S}_{\underline{\alpha}} \subseteq \mathsf{S}$ be the subsemigroup generated by $X_{\underline{\alpha}}$. Then $\mathcal{L}_{N,\underline{\alpha}} = \mathcal{T}(X_{\underline{\alpha}})/J_{\underline{\alpha}}$ where $J_{\underline{\alpha}}$ is the ideal generated by all elements of the form $[t,t]$, $t \in \mathcal{T}(X_{\underline{\alpha}})$, $[t_1,[t_2,t_3]] + [t_2,[t_3,t_1]] + [t_3,[t_2,t_1]]$, $t_1,t_2,t_3 \in \mathcal{T}(X_{\underline{\alpha}})$, and $[\alpha_{i_1},[\alpha_{i_2},\cdots,[\alpha_{i_{m-1}},\alpha_{i_m}]]\cdots]$ for any $m \leqslant N$ and $\{i_1,\ldots,i_m\} \subseteq \{1,\ldots,N\}$ such that $\alpha_{i_1} + (\alpha_{i_2} + (\cdots + \alpha_{i_m})\cdots)$ is not defined in $\mathsf{S}_{\underline{\alpha}}$. We observe in particular that $\mathcal{L}_{N,\underline{\alpha}}$ is an S–graded Lie algebra, and

$$\mathcal{L}_{N,\underline{\alpha}} = \begin{cases} \mathcal{L}_N & \text{if } \mathsf{S}^{(2)}_{\underline{\alpha}} = \mathsf{S}_{\underline{\alpha}} \times \mathsf{S}_{\underline{\alpha}} \\ \mathcal{L}^{\mathrm{ab}}_N & \text{if } \mathsf{S}^{(2)}_{\underline{\alpha}} = \emptyset \end{cases}$$

where $\mathcal{L}^{\mathrm{ab}}_N$ is the abelian Lie algebra in N generators.

By (9.5), there is a surjective map from $\prod_{\underline{\alpha} \in \mathsf{S}^N} (\mathcal{L}^{\otimes q}_{N,\underline{\alpha}})_{\delta_N}$ to $\mathsf{LA}_{\mathsf{S}}([N],[q])$. The injectivity follows easily by application of the realisation functor of LA_{S} on the S–graded Lie algebras $\mathcal{L}_{N,\underline{\alpha}}$, $\underline{\alpha} \in \mathsf{S}^N$. We then obtain an isomorphism of right $\mathsf{k}[\mathsf{S}^N] \rtimes \mathfrak{S}_N$–modules,

$$\mathsf{LA}_{\mathsf{S}}([N],[q]) \simeq \prod_{\underline{\alpha} \in \mathsf{S}^N} (\mathcal{L}^{\otimes q}_{N,\underline{\alpha}})_{\delta_N} \tag{9.7}$$

compatible with the left action of $\mathfrak{S}_q$. Through the equivalence $\mathsf{LCA}_{\mathsf{S}} \simeq \mathsf{LA}^{\mathrm{op}}_{\mathsf{S}}$, we then obtain the isomorphism of left $\mathsf{k}[\mathsf{S}^N] \rtimes \mathfrak{S}_N$–modules,

$$\underline{\mathsf{LCA}}_{\mathsf{S}}([p],[N]) \simeq \prod_{\underline{\alpha} \in \mathsf{S}^N} (\mathcal{L}^{\otimes p}_{N,\underline{\alpha}})_{\delta_N} \tag{9.8}$$

compatible with the right action of $\mathfrak{S}_p$. The following is clear.

Proposition.

(1) *The embeddings $\underline{\mathsf{LA}}_{\mathsf{S}}, \underline{\mathsf{LCA}}_{\mathsf{S}} \to \underline{\mathsf{LBA}}_{\mathsf{S}}$ induce an isomorphism of $(\mathfrak{S}_q, \mathfrak{S}_p)$–bimodules*

$$\underline{\mathsf{LBA}}_{\mathsf{S}}([p],[q]) \simeq \bigoplus_{N \geqslant 0} \underline{\mathsf{LCA}}_{\mathsf{S}}([p],[N]) \otimes_{\Gamma_{\mathsf{S},N}} \underline{\mathsf{LA}}_{\mathsf{S}}([N],[q])$$

(2) *There is an isomorphism of $(\mathfrak{S}_q,\mathfrak{S}_p)$–bimodules*

$$\underline{\mathsf{LBA}}_{\mathsf{S}}([p],[q]) \simeq \bigoplus_{N\geqslant 0}\left(\prod_{\underline{\alpha}\in\mathsf{S}^N}(\mathcal{L}_{N,\underline{\alpha}}^{\otimes p})_{\delta_N}\otimes(\mathcal{L}_{N,\underline{\alpha}}^{\otimes q})_{\delta_N}\right)_{\mathfrak{S}_N}$$

where the coinvariants are taken with respect to the diagonal action of $\mathfrak{S}_N$.

(3) *There are natural isomorphisms*

$$\underline{\mathsf{LBA}}_{\mathsf{S}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \simeq \bigoplus_{N\geqslant 0}\left(\prod_{\underline{\alpha}\in\mathsf{S}^N}(T\mathcal{L}_{N,\underline{\alpha}}^{\otimes n})_{\delta_N}\otimes(T\mathcal{L}_{N,\underline{\alpha}}^{\otimes n})_{\delta_N}\right)_{\mathfrak{S}_N}$$

$$\underline{\mathsf{LBA}}_{\mathsf{S}}(\widehat{S}[1]^{\otimes n}, S[1]^{\otimes n}) \simeq \bigoplus_{N\geqslant 0}\left(\prod_{\underline{\alpha}\in\mathsf{S}^N}(S\mathcal{L}_{N,\underline{\alpha}}^{\otimes n})_{\delta_N}\otimes(S\mathcal{L}_{N,\underline{\alpha}}^{\otimes n})_{\delta_N}\right)_{\mathfrak{S}_N}$$

In particular, if $\mathsf{S}^{(2)}=\mathsf{S}\times\mathsf{S}$, we get

$$\underline{\mathsf{LA}}_{\mathsf{S}}([N],[q]) \simeq \mathsf{k}[\mathsf{S}^N]\otimes(\mathcal{L}_N^{\otimes q})_{\delta_N} \qquad\text{and}\qquad \underline{\mathsf{LCA}}_{\mathsf{S}}([p],[N]) \simeq (\mathcal{L}_N^{\otimes p})_{\delta_N}\otimes\mathsf{k}[\mathsf{S}^N]$$

This yields the following generalisation of 6.6

$$\underline{\mathsf{LBA}}_{\mathsf{S}}([p],[q]) \simeq \bigoplus_{N\geqslant 0}\left((\mathcal{L}_N^{\otimes p})_{\delta_N}\otimes\mathsf{k}[\mathsf{S}^N]\otimes(\mathcal{L}_N^{\otimes q})_{\delta_N}\right)_{\mathfrak{S}_N}$$

9.4. **Universal Drinfeld–Yetter modules.** The category $\underline{\mathsf{DY}}_{\mathsf{S}}^n$, $n\geqslant 1$, is the colored PROP generated by $n+1$ objects $[1]$ and $[\mathsf{V}_k]$, $k=1,\ldots,n$, and morphisms

$$\mu:[2]\to[1] \qquad\qquad \delta:[1]\to[2]$$
$$\theta_\alpha:[1]\to[1],\quad \alpha\in\mathsf{S}$$
$$\pi_k:[1]\otimes[\mathsf{V}_k]\to[\mathsf{V}_k] \qquad\qquad \pi_k^*:[\mathsf{V}_k]\to[1]\otimes[\mathsf{V}_k]$$

such that $([1],\mu,\delta,\{\theta_\alpha\})$ is an $\underline{\mathsf{LBA}}_{\mathsf{S}}$–module in $\underline{\mathsf{DY}}_{\mathsf{S}}^n$, and every $([\mathsf{V}_k],\pi_k,\pi_k^*)$ is a Drinfeld–Yetter module over $[1]$.

Set

$$\mathfrak{U}_{\mathsf{S}}^n = \mathsf{End}_{\underline{\mathsf{DY}}_{\mathsf{S}}^n}\left(\bigotimes_{k=1}^n[\mathsf{V}_k]\right)$$

Let $\mathfrak{b}$ be an S–graded Lie bialgebra. Then, for any n–tuple $\{V_k,\pi_k,\pi_k^*\}_{k=1}^n$ of Drinfeld–Yetter modules over $\mathfrak{b}$, there is a realisation functor

$$\mathcal{G}_{(\mathfrak{b},V_1,\ldots,V_n)}:\underline{\mathsf{DY}}_{\mathsf{S}}^n\longrightarrow\mathsf{Vect}_{\mathsf{k}}$$

such that $[\mathfrak{b}]\mapsto\mathfrak{b}$, and $[\mathsf{V}_k]\mapsto V_k$, $k=1,\ldots,n$.

Proposition.

(1) *Let* $\mathsf{f}\colon\mathsf{DY}_{\mathfrak{b}}\to\mathsf{Vect}_{\mathsf{k}}$ *be the forgetful functor, and* $\mathcal{U}_{\mathfrak{b}}^n=\mathsf{End}\left(\mathsf{f}^{\boxtimes n}\right)$. *The functors* $\mathcal{G}_{(\mathfrak{b},V_1,\ldots,V_n)}$ *induce an algebra homomorphism*

$$\rho_{\mathsf{S},\mathfrak{b}}^n:\mathfrak{U}_{\mathsf{S}}^n\to\mathcal{U}_{\mathfrak{b}}^n$$

(2) *Set* $\mathcal{A}_{N,\underline{\alpha}}=U\mathcal{L}_{N,\underline{\alpha}}$. *There is a linear isomorphism*

$$\mathfrak{U}_{\mathsf{S}}^n \simeq \bigoplus_{N\geqslant 0}\left(\prod_{\underline{\alpha}\in\mathsf{S}^N}(\mathcal{A}_{N,\underline{\alpha}}^{\otimes n})_{\delta_N}\otimes(\mathcal{A}_{N,\underline{\alpha}}^{\otimes n})_{\delta_N}\right)_{\mathfrak{S}_N}$$

In particular, if $\mathsf{S}^{(2)} = \mathsf{S}\times\mathsf{S}$,

$$\mathfrak{U}^n_{\mathsf{S}} \simeq \bigoplus_{N\geqslant 0}\left((\mathcal{A}_N^{\otimes n})_{\delta_N}\otimes \mathsf{k}[\mathsf{S}^N]\otimes(\mathcal{A}_N^{\otimes n})_{\delta_N}\right)_{\mathfrak{S}_N}$$

(3) *Every element in* $\mathfrak{U}^n_{\mathsf{S}}$ *is a linear combination of the endomorphisms of* $[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]$ *given by*

$$r^{\underline{\alpha},\sigma}_{\underline{N},\underline{N}'} = \pi^{(\underline{N})}\circ\theta_{\underline{\alpha}}\otimes\mathrm{id}^{\otimes n}\circ\sigma\otimes\mathrm{id}^{\otimes n}\circ\pi^{*(\underline{N}')} \tag{9.9}$$

where $N\geqslant 0$, $\underline{N},\underline{N}'\in\mathbb{N}^n$ *are such that* $|\underline{N}| = N = |\underline{N}'|$, $\underline{\alpha}\in\mathsf{S}^N$, *and* $\sigma\in\mathfrak{S}_{N,\underline{\alpha}}$, *where*

$$\mathfrak{S}_{N,\underline{\alpha}} = \{\sigma\in\mathfrak{S}_N \mid (\alpha_i,\alpha_j)\notin\mathsf{S}^{(2)}, i<j\Rightarrow\sigma(i)<\sigma(j)\}$$

(4) *Let* $\{b_i\}\subset\mathfrak{b}$ *be a basis and* $\{b^i\}\subset\mathfrak{b}^*$ *the dual basis. Then one has*

$$\rho^n_{\mathsf{S}}(r^{\underline{\alpha},\sigma}_{\underline{N},\underline{N}'}) = \sum_{\substack{\underline{i}=(i_1,\ldots,i_N)\\ i_k\in I_{\alpha_k}}} b_{\underline{N}(\underline{i})}\cdot b^{\sigma(\underline{N}')(\underline{i})}$$

where I_{α_k} *is the set of indices corresponding to the basis of* $\mathfrak{b}_{\alpha_k}$.

Proof. (1) follows by construction. The proof of (2) is an easy generalisation of those in 5.3 and 5.10. Namely, we first observe that by normal ordering there is an isomorphism

$$\underline{\mathsf{DY}}_{\mathsf{S}}([\mathsf{V}_1],[\mathsf{V}_1]) \simeq \bigoplus_{N\geqslant 0}\underline{\mathsf{LCM}}_{\mathsf{S}}([\mathsf{V}_1],[N]\otimes[\mathsf{V}_1])\otimes_{\Gamma_{\mathsf{S},N}}\underline{\mathsf{LM}}_{\mathsf{S}}([N]\otimes[\mathsf{V}_1],[\mathsf{V}_1]) \tag{9.10}$$

where $\underline{\mathsf{LM}}_{\mathsf{S}}$ (resp. $\underline{\mathsf{LCM}}_{\mathsf{S}}$) is the PROP generated by an S–graded Lie algebra object $[1]$ and a $[1]$–module $[\mathsf{V}_1]$ (resp. an S–graded Lie coalgebra object $[1]$ and a $[1]$–comodule $[\mathsf{V}_1]$). By normal ordering in $\underline{\mathsf{LM}}_{\mathsf{S}}$, we obtain a surjective map

$$\prod_{\underline{\alpha}\in\mathsf{S}^N}(U\mathcal{L}_{N,\underline{\alpha}})_{\delta_N}\to\underline{\mathsf{LM}}_{\mathsf{S}}([N]\otimes[\mathsf{V}_1],[\mathsf{V}_1]) \tag{9.11}$$

which is easily seen to be an isomorphism by considering the action of the Lie algebra $\mathcal{L}_{N,\underline{\alpha}}$ on $U\mathcal{L}_{N,\underline{\alpha}}$ and the corresponding realisation functor. In particular, since $\mathsf{LCM}_{\mathsf{S}}\simeq\mathsf{LM}_{\mathsf{S}}^{\mathrm{op}}$, combining (9.10) and (9.11), we obtain a linear isomorphism

$$\mathfrak{U}_{\mathsf{S}} \simeq \bigoplus_{N\geqslant 0}\left(\prod_{\underline{\alpha}\in\mathsf{S}^N}(\mathcal{A}_{N,\underline{\alpha}})_{\delta_N}\otimes(\mathcal{A}_{N,\underline{\alpha}})_{\delta_N}\right)_{\mathfrak{S}_N}$$

The proof of the general case follows by replacing $[\mathsf{V}_1]$ with $[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]$ and $U\mathcal{L}_{N,\underline{\alpha}}$ with $U\mathcal{L}^{\otimes n}_{N,\underline{\alpha}}$. (*iii*) and (*iv*) are straightforward. □

9.5. **PBW theorem for $\mathfrak{U}^n_{\mathsf{S}}$.** As in the case of $\mathfrak{U}_{\mathsf{DY}}$ and $\mathfrak{U}_{\mathsf{PDY}}$, the tower of algebras $\{\mathfrak{U}^n_{\mathsf{S}}\}_{n\geqslant 1}$ is endowed with face maps $\Delta^n_i:\mathfrak{U}^n_{\mathsf{S}}\to\mathfrak{U}^{n+1}_{\mathsf{S}}$ and degeneration maps $\mathcal{E}^i_n:\mathfrak{U}^n_{\mathsf{S}}\to\mathfrak{U}^{n-1}_{\mathsf{S}}$ defining a cosimplicial structure.

Let

$$\mathsf{a}^n:\underline{\mathsf{LBA}}_{\mathsf{S}}(\widehat{T}[1]^{\otimes n},T[1]^{\otimes n})\to\underline{\mathsf{DY}}^n_{\mathsf{S}}(\otimes_{k=1}^n[\mathsf{V}_k],\otimes_{k=1}^n[\mathsf{V}_k])$$

be the map given on $\phi_{\underline{p},\underline{q}}\in\underline{\mathsf{LBA}}_{\mathsf{S}}(T^{\underline{p}}[1],T^{\underline{q}}[1])$, by

$$\mathsf{a}^n(\phi_{\underline{p},\underline{q}}) = \pi^{(\underline{p})}\circ\phi_{\underline{p},\underline{q}}\circ\pi^{*(\underline{q})}$$

As in the case of $\underline{\mathsf{LBA}}$ and $\underline{\mathsf{PLBA}}$, the isomorphism $\mathcal{A}_{N,\underline{\alpha}} = U\mathcal{L}_{N,\underline{\alpha}} \simeq S\mathcal{L}_{N,\underline{\alpha}}$ induces a PBW theorem for $\mathfrak{U}^n_{\mathsf{S}}$.

Theorem.

(1) *The following diagram is commutative*

$$\begin{tikzcd}
\underline{\mathsf{DY}}^n_{\mathsf{S}}(\bigotimes_{k=1}^n[\mathsf{V}_k], \bigotimes_{k=1}^n[\mathsf{V}_k]) \arrow[r] & \bigoplus_{N\geqslant 0}\left(\prod_{\underline{\alpha}\in\mathsf{S}^N}(\mathcal{A}^{\otimes n}_{N,\underline{\alpha}})_{\delta_N}\otimes(\mathcal{A}^{\otimes n}_{N,\underline{\alpha}})_{\delta_N}\right)_{\mathfrak{S}_N} \\
\underline{\mathsf{LBA}}_{\mathsf{S}}(\widehat{T}[1]^{\otimes n}, T[1]^{\otimes n}) \arrow[r] \arrow[u, "\mathsf{a}^n"] & \bigoplus_{N\geqslant 0}\left(\prod_{\underline{\alpha}\in\mathsf{S}^N}(T\mathcal{L}_{N,\underline{\alpha}}{}^{\otimes n})_{\delta_N}\otimes(T\mathcal{L}_{N,\underline{\alpha}}{}^{\otimes n})_{\delta_N}\right)_{\mathfrak{S}_N} \arrow[u] \\
\underline{\mathsf{LBA}}_{\mathsf{S}}(\widehat{S}[1]^{\otimes n}, S[1]^{\otimes n}) \arrow[r] \arrow[u, "\mathsf{Sym}"] & \bigoplus_{N\geqslant 0}\left(\prod_{\underline{\alpha}\in\mathsf{S}^N}(S\mathcal{L}_{N,\underline{\alpha}}{}^{\otimes n})_{\delta_N}\otimes(S\mathcal{L}_{N,\underline{\alpha}}{}^{\otimes n})_{\delta_N}\right)_{\mathfrak{S}_N} \arrow[u, "\mathsf{Sym}\otimes\mathsf{Sym}"']
\end{tikzcd}$$

(2) *The map $\mathsf{a}^n\circ\mathsf{Sym}$ is an isomorphism of cosimplicial spaces.*

9.6. **Hochschild cohomology.** The cosimplicial structure on $\{\mathfrak{U}^n_{\mathsf{S}}\}_{n\geqslant 1}$ gives rise to the *semigroup* universal Hochschild complex with differential $d^n_{\mathsf{S}} = \sum_{i=0}^{n+1}(-1)^i\Delta^n_i : \mathfrak{U}^n_{\mathsf{S}} \to \mathfrak{U}^{n+1}_{\mathsf{S}}$ The morphisms $\{\rho^n_{\mathsf{S},\mathfrak{b}}\}_{n\geqslant 1}$ defined in 9.4 define a chain map between the corresponding Hochschild complexes. As in 5.19 and 6.11, we get the following

Theorem.

(1) *The map $\mathsf{a}^n\circ\mathsf{Sym}$ induces an isomorphism*

$$H^i(\mathfrak{U}^\bullet_{\mathsf{S}}, d_{\mathsf{S}}) \cong \bigoplus_{j=0}^{i}\underline{\mathsf{LBA}}_{\mathsf{S}}(\wedge^j[1], \wedge^{i-j}[1])$$

In particular, $H^0(\mathfrak{U}^\bullet_{\mathsf{S}}, d_{H,\mathsf{S}}) = \mathsf{k}$ and $H^1(\mathfrak{U}^\bullet_{\mathsf{S}}, d_{\mathsf{S}}) = 0$.

(2) *The identification in terms of semigroup Lie algebras $\mathcal{L}_{N,\underline{\alpha}}$ of Proposition 9.3 yields*

$$H^i(\mathfrak{U}^\bullet_{\mathsf{S}}, d_H) \cong \bigoplus_{N\geqslant 0}\bigoplus_{j=0}^{i}\left(\prod_{\underline{\alpha}\in\mathsf{S}^N}\left(\wedge^j\mathcal{L}_{N,\underline{\alpha}}\right)_{\delta_N}\otimes\left(\wedge^{i-j}\mathcal{L}_{N,\underline{\alpha}}\right)_{\delta_N}\right)_{\mathfrak{S}_N}$$

9.7. **Gluing maps.** If $\mathfrak{b}$ is a Lie bialgebra with Drinfeld double $\mathfrak{g}_{\mathfrak{b}}$, the standard multiplication maps $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n} \to U\mathfrak{g}_{\mathfrak{b}}^{\otimes n-1}$ cannot be lifted to the PROPic level since this would imply, for example, that the anti–normally ordered Casimir $m(r_{21}) = \sum_i b^i b_i$ acts on any Drinfeld–Yetter module, which is not the case if $\mathfrak{b}$ is infinite–dimensional. However, the *polarised* multiplication maps $U(\mathfrak{b}^*)^{\otimes n}\otimes U\mathfrak{b}^{\otimes n} \to U(\mathfrak{b}^*)^{\otimes n-1}\otimes U\mathfrak{b}^{\otimes n-1}$ do admit a universal analogue as maps from $\mathfrak{U}^n_{\mathsf{S}}$ to $\mathfrak{U}^{n-1}_{\mathsf{S}}$. Their description in terms of associative algebras, under the identification given by Proposition 9.4, simply corresponds to polarised multiplication maps $\mathcal{A}^{\otimes n}_{N,\underline{\alpha}}\otimes\mathcal{A}^{\otimes n}_{N,\underline{\alpha}} \to \mathcal{A}^{\otimes n-1}_{N,\underline{\alpha}}\otimes\mathcal{A}^{\otimes n-1}_{N,\underline{\alpha}}$ (cf. [12, Prop. A.1]).

Their intrinsic description in terms of morphisms in $\underline{\mathsf{DY}}^n_{\mathsf{S}}$, however, is more involved. Roughly, we consider maps $m^i_n : \mathfrak{U}^n_{\mathsf{S}} \to \mathfrak{U}^{n-1}_{\mathsf{S}}$, $i = 1, \dots, n-1$, which produce an endomorphism of $[\mathsf{V}_1] \otimes \cdots \otimes [\mathsf{V}_{n-1}]$ from one of $[\mathsf{V}_1] \otimes \cdots \otimes [\mathsf{V}_n]$ by *gluing* together the modules $[\mathsf{V}_i]$ and $[\mathsf{V}_{i+1}]$, as we now describe. This definition relies on the universal Verma modules $[M]$ and $[M^\vee]$ constructed in [15] (see also [1, Sec. 4] for more details). As objects in $\underline{\mathsf{LBA}}_{\mathsf{S}}$, $[M] = S[1]$, $[M^\vee] = \widehat{S}[1] \coloneqq \prod_{N \geqslant 0} S^N[1]$. They are endowed with a structure of Drinfeld–Yetter module over [1], and they satisfy

$$\pi_{[M]} \circ \mathrm{id}_{[1]} \otimes \iota = i_{[1]} \qquad \text{and} \qquad \mathrm{id}_{[1]} \otimes \varepsilon \circ \pi^*_{[M^\vee]} = p_{[1]}$$

where ι and $i_{[1]}$ (resp. ε and $p_{[1]}$) are the canonical injections of (resp. projections to) $[0] = S^0[1]$ and $[1] = S^1[1]$.

9.7.1. Let $\underline{\mathsf{LM}}_{\mathsf{S}}$ be the PROP generated by an S–graded Lie algebra object [1] and [1]–modules $[\mathsf{V}_k]$, $k = 1, \dots, n$. Let $\pi_k : [M] \otimes [\mathsf{V}_k] \to [\mathsf{V}_k]$ be the map obtained by iterations of the action of [1] on $[\mathsf{V}_k]$.

Definition. For any $n \geqslant 2$, $i = 1, \dots, n-1$, the nth *action–gluing* map in position i

$$(m_{\underline{\mathsf{LM}}})^i_n : \underline{\mathsf{LM}}_{\mathsf{S}} \left([N] \otimes \bigotimes_{k=1}^{n} [\mathsf{V}_k], \bigotimes_{k=1}^{n} [\mathsf{V}_k] \right) \to \underline{\mathsf{LM}}_{\mathsf{S}} \left([N] \otimes \bigotimes_{k=1}^{n-1} [\mathsf{V}_k], \bigotimes_{k=1}^{n-1} [\mathsf{V}_k] \right)$$

is defined by

$$(m_{\underline{\mathsf{LM}}})^i_n \left(\phi_{[N],[\mathsf{V}_1],\dots,[\mathsf{V}_n]} \right) = \pi^{(i)}_i \circ \phi_{[N],[\mathsf{V}_1],\dots,[\mathsf{V}_{i-1}],[M],[\mathsf{V}_i],\dots,[\mathsf{V}_{n-1}]} \circ \iota^{(i)}$$

where $\pi^{(i)}_i = \mathrm{id}_{[\mathsf{V}_{[1,i-1]}]} \otimes \pi_i \otimes \mathrm{id}_{[\mathsf{V}_{[i+1,n-1]}]}$, $\iota^{(i)} = \mathrm{id}_{[N]} \otimes \mathrm{id}_{[\mathsf{V}_{[1,i-1]}]} \otimes \iota \otimes \mathrm{id}_{[\mathsf{V}_{[i,n-1]}]}$, and $\mathrm{id}_{[\mathsf{V}_{[i,j]}]} \coloneqq \mathrm{id}_{[\mathsf{V}_i] \otimes \cdots \otimes [\mathsf{V}_j]}$, $i \leqslant j$.

9.7.2. Similarly, let $\underline{\mathsf{LCM}}^n_{\mathsf{S}}$ be the PROP generated by an S–graded Lie coalgebra object [1] and [1]–comodules $[\mathsf{V}_k]$, $k = 1, \dots, n$. Let $\pi^*_k : [\mathsf{V}_k] \to [M^\vee] \otimes [\mathsf{V}_k]$ be the map obtained by iterations of the coaction of [1] on $[\mathsf{V}_k]$.

Definition. For any $n \geqslant 2$, $i = 1, \dots, n-1$, the nth *coaction–gluing* map in position i

$$(m_{\underline{\mathsf{LCM}}})^i_n : \underline{\mathsf{LCM}}_{\mathsf{S}} \left(\bigotimes_{k=1}^{n} [\mathsf{V}_k], [N] \otimes \bigotimes_{k=1}^{n} [\mathsf{V}_k] \right) \to \underline{\mathsf{LCM}}_{\mathsf{S}} \left(\bigotimes_{k=1}^{n-1} [\mathsf{V}_k], [N] \otimes \bigotimes_{k=1}^{n-1} [\mathsf{V}_k] \right)$$

is defined by

$$(m_{\underline{\mathsf{LCM}}})^i_n \left(\phi_{[N],[\mathsf{V}_1],\dots,[\mathsf{V}_n]} \right) = \varepsilon^{(i)} \circ \phi_{[N],[\mathsf{V}_1],\dots,[\mathsf{V}_{i-1}],[M],[\mathsf{V}_i],\dots,[\mathsf{V}_{n-1}]} \circ (\pi^*_i)^{(i)}$$

where $(\pi^*_i)^{(i)} = \mathrm{id}_{[\mathsf{V}_{[1,i-1]}]} \otimes \pi^*_i \otimes \mathrm{id}_{[\mathsf{V}_{[i+1,n-1]}]}$, $\varepsilon^{(i)} = \mathrm{id}_{[N]} \otimes \mathrm{id}_{[\mathsf{V}_{[1,i-1]}]} \otimes \varepsilon \otimes \mathrm{id}_{[\mathsf{V}_{[i,n-1]}]}$.

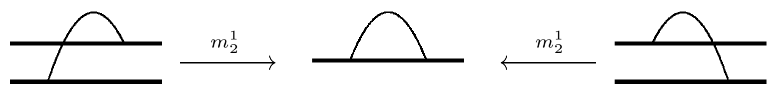
9.7.3. Set $[\mathsf{V}_{[1,n]}] = \bigotimes_{k=1}^n [\mathsf{V}_k]$. As we observed in (9.10), by normal ordering, the algebra of endomorphisms $\mathfrak{U}^n_{\mathsf{S}} = \underline{\mathsf{DY}}^n_{\mathsf{S}} \left([\mathsf{V}_{[1,n]}], [\mathsf{V}_{[1,n]}] \right)$ is isomorphic to

$$\bigoplus_{N \geqslant 0} \underline{\mathsf{LCM}}_{\mathsf{S}} \left([\mathsf{V}_{[1,n]}], [N] \otimes [\mathsf{V}_{[1,n]}] \right) \otimes_{\Gamma_{\mathsf{S},N}} \underline{\mathsf{LM}}_{\mathsf{S}} \left([N] \otimes [\mathsf{V}_{[1,n]}], [\mathsf{V}_{[1,n]}] \right)$$

where $\Gamma_{\mathsf{S},N} = \mathsf{k}[\mathsf{S}^N] \rtimes \mathfrak{S}_N$.

Definition. For any $n \geq 1$, $i = 1, \dots, n$, the nth gluing map in position i, $m^i_n : \mathfrak{U}^n_{\mathsf{S}} \to \mathfrak{U}^{n-1}_{\mathsf{S}}$, is the map induced by $(m_{\underline{\mathsf{LCM}}})^i_n \otimes (m_{\underline{\mathsf{LM}}})^i_n$.

Remark. It should be clear from the description above that the maps m_n^i reduce an element of $\mathfrak{U}_{\mathsf{S}}^n$ to one of $\mathfrak{U}_{\mathsf{S}}^{n-1}$ by *gluing* together $[\mathsf{V}_i]$ and $[\mathsf{V}_{i+1}]$, preserving the order of actions and coactions. Specifically, the coactions on $[\mathsf{V}_i]$ occur before (resp. any action on $[\mathsf{V}_i]$ occur after) any coaction or action on $[\mathsf{V}_{i+1}]$. As we anticipated, these should not be mistaken with a universal version of the multiplication maps $U\mathfrak{g}^{\otimes n} \to U\mathfrak{g}^{\otimes n-1}$. For example, let $r_{\underline{N},\underline{N}'}^{\underline{\alpha},\sigma}$ be as in 9.9 and set $r_{12}^{\alpha} = r_{(0,1),(1,0)}^{\alpha,\mathrm{id}}$, $r_{21}^{\alpha} = r_{(1,0),(0,1)}^{\alpha,\mathrm{id}}$, and $\kappa_\alpha = r_{1,1}^{\alpha,\mathrm{id}}$. Then, $m_2^1(r_{12}^\alpha) = \kappa_\alpha = m_2^1(r_{21}^\alpha)$ or, pictorially,



9.8. **The Casimir operator of $\mathfrak{U}_{\mathsf{DY}}$.** Recall that $\kappa = r_{1,1}^{\mathrm{id}} \in \mathfrak{U}_{\mathsf{DY}}^1$ is the universal version of the normally ordered Casimir $\sum_i b_i b^i$. In $\mathfrak{U}_{\mathsf{S}}^1$, we have $\widetilde{\rho}_{\mathsf{S}}(\kappa) = \sum_{\alpha\in\mathsf{S}} \kappa_\alpha$. For any $i = 1, \dots, n$, set

$$\kappa_i = \mathrm{id}_{[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_{i-1}]} \otimes \kappa \otimes \mathrm{id}_{[\mathsf{V}_{i+1}]\otimes\cdots\otimes[\mathsf{V}_n]} \in \mathfrak{U}_{\mathsf{S}}^n$$

Proposition. *The element $\sum_{i=1}^n (\kappa)_i$ is central in $\mathfrak{U}_{\mathsf{S}}^n$.*

Remark. Note that in the algebra $\mathfrak{U}_{\mathsf{S}}$ the notions of invariant and central element stand in an opposite relation than they do for the algebra $U\mathfrak{g}_{\mathfrak{b}}$. Namely, an invariant element is clearly central, but the opposite in not necessarily true. For example the Casimir element κ is central but not invariant.

Proof. The argument below is an easy generalisation of [12, Prop. A.1]. Let $\mathcal{X}_n \subseteq \mathfrak{U}_{\mathsf{S}}^n$ be the subspace of all elements $X \in \mathfrak{U}_{\mathsf{S}}^n$ satisfying

$$[\kappa_1 + \kappa_2 + \cdots + \kappa_n, X] = 0$$

The following is straightforward.

(i) If $X \in \mathcal{X}_n$, then $\Delta_i^n(X) \in \mathcal{X}_{n+1}$, $i = 0, 1, \dots, n+1$.
(ii) If $X \in \mathcal{X}_n$, then $X^\sigma \in \mathcal{X}_n$ for any $\sigma \in \mathfrak{S}_n$, where

$$X^\sigma = \sigma^{-1} \circ X_{[\mathsf{V}_{\sigma(1)}]\otimes\cdots\otimes[\mathsf{V}_{\sigma(n)}]} \circ \sigma$$

(iii) If $X \in \mathcal{X}_n$, then $m_n^i(X) \in \mathcal{X}_{n-1}$ for any $n \geqslant 2$ and $i = 1, \dots, n-1$.

To prove (iii) it is enough to observe that, for any $X \in \mathfrak{U}_{\mathsf{S}}^n$,

$$m_n^i([X, \kappa_1 + \kappa_2 + \cdots + \kappa_n]) = [m_n^i(X), \kappa_1 + \kappa_2 + \cdots + \kappa_{n-1}]$$

Let $\mathcal{P}_N$ be the set of partitions of $\{1, \dots, 2N\}$ of the form $\{i_1, j_1\} \sqcup \cdots \{i_N, j_N\}$. For any partition P in $\mathcal{P}_N$ and $\underline{\alpha} \in \mathsf{S}^N$, set

$$r_P^{\underline{\alpha}} = \prod_{k=1}^N r_{i_k,j_k}^{\alpha_k}$$

where $r_{i_k,j_k}^{\alpha_k}$ denotes the composition of the coaction on $[\mathsf{V}_{j_k}]$, the idempotent θ_{α_k}, and the action on $[\mathsf{V}_{i_k}]$. The morphisms $r_P^{\underline{\alpha}}$ are well–defined, since the elements $r_{i_k,j_k}^{\alpha_k}$, $k = 1, \dots, N$, commute in $\mathfrak{U}_{\mathsf{S}}^{2n}$. It follows from $(i), (ii), (iii)$ that $\mathcal{X}_n = \mathfrak{U}_{\mathsf{S}}^n$ for any $n \geqslant 1$ if and only if

$$\{r_P^{\underline{\alpha}} \mid P \in \mathcal{P}_n, \underline{\alpha} \in \mathsf{S}^n\} \subset \mathcal{X}_{2n}$$

for all $n \geqslant 1$. The result follows from the explicit computation $[r_{12}^\alpha, \kappa_1 + \kappa_2] = 0$, for any $\alpha \in \mathsf{S}$.

Namely, for any $\beta \in \mathsf{S}$, we have

$$[r_{12}^{\alpha}, (\kappa_\beta)_1 + (\kappa_\beta)_2] = C_{\alpha,\beta} - \sum_{\gamma \in L(\beta)} C_{\alpha,\gamma}$$

where $L(\beta) = \{\gamma \in \mathsf{S} \mid \alpha + \gamma = \beta\}$ and

$$C_{\alpha,\beta} = r^{(\beta,\alpha),\mathrm{id}}_{(1,1),(2,0)} - r^{(\beta,\alpha),(1\,2)}_{(1,1),(2,0)} + r^{(\alpha,\beta),\mathrm{id}}_{(0,2),(1,1)} - r^{(\beta,\alpha),(1\,2)}_{(0,2),(1,1)}$$

Set $A_\alpha = \{\beta \in \mathsf{S} \mid L(\beta) \neq \emptyset\}$, $B_\alpha = \{\beta \in \mathsf{S} \mid (\alpha, \beta) \in \mathsf{S}^{(2)}\}$, and $L(A_\alpha) = \bigsqcup_{\beta \in A_\alpha} L(\beta)$. It is clear that, if $\beta \notin B_\alpha$, then $C_{\alpha,\beta} = 0$. Therefore,

$$\begin{aligned} [r_{12}^{\alpha}, \kappa_1 + \kappa_2] &= \sum_{\beta \in \mathsf{S}} \left(C_{\alpha,\beta} - \sum_{\gamma \in L(\beta)} C_{\alpha,\gamma} \right) = \sum_{\beta \in B_\alpha} C_{\alpha,\beta} - \sum_{\beta \in A_\alpha} \sum_{\gamma \in L(\beta)} C_{\alpha,\gamma} \\ &= \sum_{\beta \in B_\alpha} C_{\alpha,\beta} - \sum_{\gamma \in L(A_\alpha)} C_{\alpha,\gamma} \end{aligned}$$

The result follows by observing that, if $\beta \in B_\alpha$, then $\beta \in L(\alpha + \beta) \subset L(A_\alpha)$. Therefore, $B_\alpha = L(A_\alpha)$ and $[r_{12}^{\alpha}, \kappa_1 + \kappa_2] = 0$. $\square$

10. Saturated subsemigroups and split pairs

Let S be a semigroup, and $\mathfrak{U}^n_{\mathsf{S}}$ the universal algebras introduced in 9.4. In this section, we study the subalgebras of $\mathfrak{U}^n_{\mathsf{S}}$ determined by the saturated subsemigroups of S.

10.1. **Subsemigroups and $\underline{\mathsf{LBA}}$–modules in $\underline{\mathsf{LBA}}_{\mathsf{S}}$.** Recall from 8.3 that a subsemigroup $\mathsf{S}' \subseteq \mathsf{S}$ is saturated if $\mathsf{S}^{(2)}_\alpha \subseteq \mathsf{S}' \times \mathsf{S}'$ for any $\alpha \in \mathsf{S}'$.

Proposition. *Let $\mathsf{S}' \subseteq \mathsf{S}$ be a saturated subsemigroup in S. Then,*

(1) *The idempotent $\theta_{\mathsf{S}'} = \sum_{\alpha \in \mathsf{S}'} \theta_\alpha$ satisfies*

$$\theta_{\mathsf{S}'} \circ \mu = \mu \circ \theta_{\mathsf{S}'} \otimes \theta_{\mathsf{S}'} \qquad \text{and} \qquad \delta \circ \theta_{\mathsf{S}'} = \theta_{\mathsf{S}'} \otimes \theta_{\mathsf{S}'} \circ \delta$$

(2) *The object $([1]_{\mathsf{S}'}, \mu_{\mathsf{S}'}, \delta_{\mathsf{S}'})$, where $[1]_{\mathsf{S}'} = ([1], \theta_{\mathsf{S}'})$,*

$$\mu_{\mathsf{S}'} = \theta_{\mathsf{S}'} \circ \mu \circ \theta_{\mathsf{S}'} \otimes \theta_{\mathsf{S}'} \qquad \text{and} \qquad \delta_{\mathsf{S}'} = \theta_{\mathsf{S}'} \otimes \theta_{\mathsf{S}'} \circ \delta \circ \theta_{\mathsf{S}'}$$

is an $\underline{\mathsf{LBA}}$–module in $\underline{\mathsf{LBA}}_{\mathsf{S}}$ and a Lie subbialgebra of $[1]$.

(3) *For any saturated subsemigroup $\mathsf{S}'' \subseteq \mathsf{S}' \subseteq \mathsf{S}$, the pair $([1]_{\mathsf{S}'}, [1]_{\mathsf{S}''})$ is a $\underline{\mathsf{PLBA}}$–module in $\underline{\mathsf{LBA}}_{\mathsf{S}}$,i.e., there is a canonical functor*

$$\rho_{(\mathsf{S}',\mathsf{S}'')} : \underline{\mathsf{PLBA}} \to \underline{\mathsf{LBA}}_{\mathsf{S}}$$

mapping $[\mathfrak{a}]$ to $[1]_{\mathsf{S}''}$ and $[\mathfrak{b}]$ to $[1]_{\mathsf{S}'}$.

Proof. We prove (1). Then (2) and (3) are obvious consequences. We have

$$\begin{aligned} \theta_{\mathsf{S}'} \circ \mu = \sum_{\alpha \in \mathsf{S}'} \theta_\alpha \circ \mu &= \sum_{\substack{\alpha \in \mathsf{S}' \\ (\beta,\gamma) \in \mathsf{S}^{(2)}_\alpha}} \mu \circ \theta_\beta \otimes \theta_\gamma = \sum_{\substack{\alpha \in \mathsf{S}' \\ (\beta,\gamma) \in \mathsf{S}'^{(2)}_\alpha}} \mu \circ \theta_\beta \otimes \theta_\gamma \\ &= \sum_{(\beta,\gamma) \in \mathsf{S}'^{(2)}} \mu \circ \theta_\beta \otimes \theta_\gamma = \mu \circ \theta_{\mathsf{S}'} \otimes \theta_{\mathsf{S}'} \end{aligned}$$

where the second equality holds by Lemma 9.1.2, the third one by saturation of S', and the fourth one because S' is a semigroup. $\square$

10.2. **Semigroup subalgebras.** Let $\mathsf{S}'\subseteq\mathsf{S}$ be a saturated subsemigroup, $\underline{\mathsf{DY}}^n_{\mathsf{S}'}$, $\underline{\mathsf{DY}}^n_{\mathsf{S}}$ the corresponding PROPs.

For every $n\geqslant 1$ and $i=0,1,\dots,n+1$, let $\mathcal{D}^n_{i,\mathsf{S}}:\underline{\mathsf{DY}}^n_{\mathsf{S}}\to\underline{\mathsf{DY}}^{n+1}_{\mathsf{S}}$ and $\mathcal{E}^{i,\mathsf{S}}_n:\underline{\mathsf{DY}}^n_{\mathsf{S}}\to\underline{\mathsf{DY}}^{n-1}_{\mathsf{S}}$ be the S–analogues of the functors $\mathcal{D}^n_i$, $\mathcal{E}^{(i)}_n$ defined in 5.7. The following is straightforward.

Proposition.

(1) *The inclusion $\mathsf{S}'\subset\mathsf{S}$ induces a faithful functor $\iota^n_{\mathsf{S}',\mathsf{S}}:\underline{\mathsf{DY}}^n_{\mathsf{S}'}\to\underline{\mathsf{DY}}^n_{\mathsf{S}}$ defined as follows: the S'–graded Lie bialgebra object $([1]_{\mathsf{S}'},\mu_{\mathsf{S}'},\delta_{\mathsf{S}'},\{\theta_\alpha\}_{\alpha\in\mathsf{S}'})$ in $\underline{\mathsf{DY}}^n_{\mathsf{S}'}$ maps to*

$$(([1]_{\mathsf{S}},\theta_{\mathsf{S}'}),\theta_{\mathsf{S}'}\circ\mu_{\mathsf{S}}\circ\theta_{\mathsf{S}'}\otimes\theta_{\mathsf{S}'},\theta_{\mathsf{S}'}\otimes\theta_{\mathsf{S}'}\circ\delta_{\mathsf{S}}\circ\theta_{\mathsf{S}'},\{\theta_\alpha\}_{\alpha\in\mathsf{S}'})$$

and the Drinfeld–Yetter modules $[\mathsf{V}_k]$ in $\underline{\mathsf{DY}}^n_{\mathsf{S}'}$ map to their analogues in $\underline{\mathsf{DY}}^n_{\mathsf{S}}$ restricted to $([1]_{\mathsf{S}},\theta_{\mathsf{S}'})$, i.e.,

$$([\mathsf{V}_k],\pi_{k,\mathsf{S}'},\pi^*_{k,\mathsf{S}'})\mapsto([\mathsf{V}_k],\pi_{k,\mathsf{S}}\circ\theta_{\mathsf{S}'}\otimes\mathrm{id}_{[\mathsf{V}_k]},\theta_{\mathsf{S}'}\otimes\mathrm{id}_{[\mathsf{V}_k]}\circ\pi^*_{k,\mathsf{S}})$$

(2) *For any $i=0,\dots,n+1$,*

$$\iota^{n+1}_{\mathsf{S}',\mathsf{S}}\circ\mathcal{D}^n_{i,\mathsf{S}'}=\mathcal{D}^n_{i,\mathsf{S}}\circ\iota^n_{\mathsf{S}',\mathsf{S}}\qquad\text{and}\qquad\iota^{n-1}_{\mathsf{S}',\mathsf{S}}\circ\mathcal{E}^{i,\mathsf{S}'}_n=\mathcal{E}^{i,\mathsf{S}}_n\circ\iota^n_{\mathsf{S}',\mathsf{S}}$$

It follows from Theorem 9.5 that the faithful functor $\iota^n_{\mathsf{S}',\mathsf{S}}$ induce an injective morphism of algebras $\mathfrak{U}^n_{\mathsf{S}'}\to\mathfrak{U}^n_{\mathsf{S}}$, which preserves the cosimplicial structures induced on $\mathfrak{U}^n_{\mathsf{S}'}$ and $\mathfrak{U}^n_{\mathsf{S}}$ by the functors $\mathcal{D}^n_{i,\cdot},\mathcal{E}^{i,\cdot}_n$. Henceforth, for every subsemigroup S', we will identify the algebras $\mathfrak{U}^n_{\mathsf{S}'}$ with their images in $\mathfrak{U}^n_{\mathsf{S}}$.

10.3. **Subsemigroup invariants.** Let $\mathsf{S}'\subseteq\mathsf{S}$ be a subsemigroup. If $\mathfrak{b}$ is an S–graded Lie bialgebra, the subspace

$$\mathfrak{b}_{\mathsf{S}'}=\bigoplus_{\alpha\in\mathsf{S}'}\mathfrak{b}_\alpha\subseteq\mathfrak{b}\tag{10.1}$$

is a Lie subalgebra of $\mathfrak{b}$, and it is a Lie subbialgebra if S' is saturated. Denote by

$$\mathcal{U}^n_{\mathfrak{b},\mathfrak{b}_{\mathsf{S}'}}=\mathsf{End}\left(\mathsf{DY}_{\mathfrak{b}}\to\mathsf{DY}_{\mathfrak{b}_{\mathsf{S}'}}\right)$$

the algebra of endomorphisms of the restriction functor from the category of Drinfeld–Yetter modules over $\mathfrak{b}$ to those over $\mathfrak{b}_{\mathsf{S}'}$.

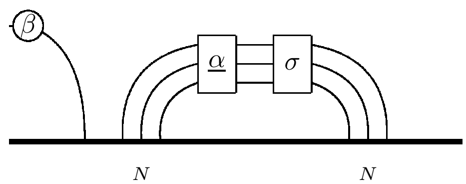
Let $[\mathfrak{b}_{\mathsf{S}'}]=([1],\theta_{\mathsf{S}'})\in\underline{\mathsf{LBA}}_{\mathsf{S}}$ be the $\underline{\mathsf{LBA}}$–module corresponding to S', and $\mathfrak{U}^n_{\mathsf{S},\mathsf{S}'}:=(\mathfrak{U}^n_{\mathsf{S}})^{[\mathfrak{b}_{\mathsf{S}'}]}\subseteq\mathfrak{U}^n_{\mathsf{S}}$ the subalgebra of $[\mathfrak{b}_{\mathsf{S}'}]$–invariants, *i.e.*, the subspace of all $\phi\in\mathfrak{U}^n_{\mathsf{S}}$ which commute with the action and coaction of $[\mathfrak{b}_{\mathsf{S}'}]$ on $[\mathsf{V}_1]\otimes\cdots\otimes[\mathsf{V}_n]$ (cf. 6.8).

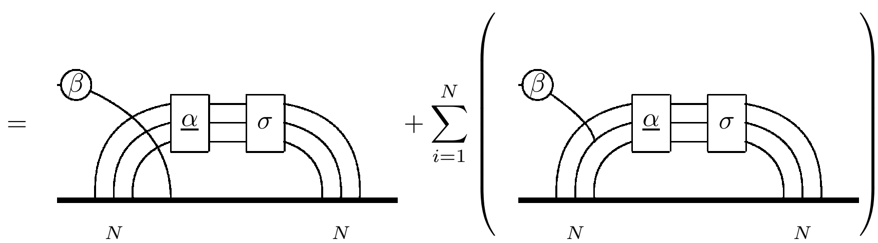
Proposition.

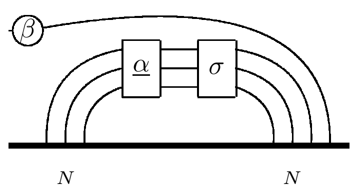
(1) *For any $\underline{\alpha}\in(\mathsf{S}\setminus\mathsf{S}')^N$, the elements $r^{\underline{\alpha},\sigma}_{\underline{N},\underline{N'}}\in\mathfrak{U}^n_{\mathsf{S}}$ given by (9.9) are invariant under $[\mathfrak{b}_{\mathsf{S}'}]$.*

(2) *If $(\mathsf{S}')^{(2)}=\emptyset$, then $[\mathfrak{b}_{\mathsf{S}'}]$ is an abelian Lie bialgebra and $\mathfrak{U}^n_{\mathsf{S},\mathsf{S}'}=\mathfrak{U}^n_{\mathsf{S}}$. In particular, the homomorphism $\rho^n_{\mathsf{S},\mathfrak{b}}:\mathfrak{U}^n_{\mathsf{S}}\to\mathcal{U}^n_{\mathfrak{b}}$ factors through $\mathcal{U}^n_{\mathfrak{b},\mathfrak{b}_{\mathsf{S}'}}$.*

Proof. (1) It follows from (9.2) and (9.4) that, for any $\underline{\alpha}\in\mathsf{S}^N$ with $\alpha_i\notin\mathsf{S}'$, $i=1,\dots,N$, the elements $r_{\underline{N},\underline{N}'}^{\underline{\alpha},\sigma}$ commutes with the action and coaction of $[\mathfrak{b}_{\mathsf{S}'}]$. We give the proof of the invariance with respect to the action of $[\mathfrak{b}_{\mathsf{S}'}]$ for the elements $r_{N,N}^{\underline{\alpha},\sigma}$. The general case is proved similarly. For any $\beta\in\mathsf{S}'$, we have





$=$ 

The identity is obtained by iteration of the compatibility condition between action and coaction, and by observing that, since S' is saturated and $\alpha_i\notin\mathsf{S}'$, one has

$$\theta_{\alpha_i}\otimes\mathrm{id}\circ\delta\circ\theta_\beta=0\qquad\text{and}\qquad\theta_\beta\circ\mu\circ\theta_{\alpha_i}\otimes\mathrm{id}=0$$

from (9.2) and (9.4). Therefore $r_{\underline{N},\underline{N}'}^{\underline{\alpha},\sigma}\in(\mathfrak{U}_{\mathsf{S}}^n)^{[\mathfrak{b}_0]}$ for any $\underline{\alpha}\in(\mathsf{S}\setminus\mathsf{S}')^N$.

(2) If $(\mathsf{S}')^{(2)}=\emptyset$, then $[\mathfrak{b}_{\mathsf{S}'}]\subseteq[\mathfrak{b}]$ is an abelian Lie subbialgebra and the same proof works for any $\underline{\alpha}\in\mathsf{S}^N$. Therefore every morphisms $\rho_{\mathsf{S},\mathfrak{b}}^n(r_{\underline{N},\underline{N}'}^{\underline{\alpha},\sigma})$ is a morphism in the category of Drinfeld–Yetter $\mathfrak{b}_{\mathsf{S}'}$–modules, and $\rho_{\mathsf{S},\mathfrak{b}}^n$ factors through $\mathcal{U}_{\mathfrak{b},\mathfrak{b}_{\mathsf{S}'}}^n$. $\square$

10.4. **Semigroup subalgebras of the S–universal algebra.** Let $\mathsf{S}''\subseteq\mathsf{S}'\subseteq\mathsf{S}$ be saturated subsemigroups, $\mathfrak{b}$ an S–graded Lie bialgebra, and $\mathfrak{b}_{\mathsf{S}''}\subseteq\mathfrak{b}_{\mathsf{S}'}\subseteq\mathfrak{b}_{\mathsf{S}}=\mathfrak{b}$ the sub Lie bialgebras defined by (10.1). For any $n\geqslant 1$, we denote by $\rho_{(\mathsf{S}',\mathsf{S}'')}^n$ the morphism $\rho_{(\mathsf{S}',\mathsf{S}'')}^n:\mathfrak{U}_{\mathsf{PDY}}^n\to\mathcal{U}_{\mathfrak{b}}^n$ corresponding to the split pair of Lie bialgebras $(\mathfrak{b}_{\mathsf{S}'},\mathfrak{b}_{\mathsf{S}''})$. The following is clear.

Proposition.

(1) *Let* $\widetilde{\rho}^n_{\mathsf{S}'}:\mathfrak{U}^n_{\mathsf{DY}}\to\mathfrak{U}^n_{\mathsf{S}}$ *be the linear map which is the identity on each* $[\mathsf{V}_k]$, *and maps the Lie bialgebra* $[1]\in\underline{\mathsf{DY}}^n$ *to* $([1],\theta_{\mathsf{S}'})\in\underline{\mathsf{DY}}^n_{\mathsf{S}}$*. In particular,*

$$\widetilde{\rho}^n_{\mathsf{S}'}(r^{\sigma}_{\underline{N},\underline{N}'})=\sum_{\underline{\alpha}\in(\mathsf{S}')^N}r^{\underline{\alpha},\sigma}_{\underline{N},\underline{N}'}$$

Then, $\widetilde{\rho}_{\mathsf{S}'}$ *is an algebra homomorphism and satisfies*

$$\begin{tikzcd}
\mathfrak{U}^n_{\mathsf{DY}} \arrow[r, "\widetilde{\rho}^n_{\mathsf{S}'}"] \arrow[d, "\rho^n_{\mathfrak{b}_{\mathsf{S}'}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{U}^n_{\mathsf{S}} \arrow[d, "\rho^n_{\mathsf{S}}"] \\
\mathcal{U}^n_{\mathfrak{b}_{\mathsf{S}'}} \arrow[r] & \mathcal{U}^n_{\mathfrak{b}_{\mathsf{S}}}
\end{tikzcd}$$

(2) *Let* $\widetilde{\rho}^n_{(\mathsf{S}',\mathsf{S}'')}:\mathfrak{U}^n_{\mathsf{PDY}}\to\mathfrak{U}^n_{\mathsf{S}}$ *be the linear map which is the identity on each* $[\mathsf{V}_k]$*, and maps the split pair* $([1],\mathrm{id}),([1],\pi_0)$ *in* $\underline{\mathsf{PDY}}^n$ *to the split pair* $([1],\theta_{\mathsf{S}'}),([1],\theta_{\mathsf{S}''})$ *in* $\underline{\mathsf{DY}}^n_{\mathsf{S}}$*. In particular,*

$$\widetilde{\rho}^n_{(\mathsf{S}',\mathsf{S}'')}(r^{\underline{i},\sigma}_{\underline{N},\underline{N}'})=\sum_{\underline{\alpha}\in\mathcal{I}^{\underline{i}}_{(\mathsf{S}',\mathsf{S}'')}}r^{\underline{\alpha},\sigma}_{\underline{N},\underline{N}'}$$

where $\underline{\alpha}\in\mathsf{S}^N$ *belongs to* $\mathcal{I}^{\underline{i}}_{(\mathsf{S}',\mathsf{S}'')}$ *iff* $\alpha_k\in\mathsf{S}''$ *whenever* $i_k=0$ *and* $\alpha_k\in\mathsf{S}'\backslash\mathsf{S}''$ *otherwise. Then,* $\widetilde{\rho}^n_{(\mathsf{S}',\mathsf{S}'')}$ *is an algebra homomorphism and satisfies*

$$\begin{tikzcd}
\mathfrak{U}^n_{\mathsf{PDY}} \arrow[r, "\widetilde{\rho}^n_{(\mathsf{S}',\mathsf{S}'')}"] \arrow[d, "\rho^n_{(\mathfrak{b}_{\mathsf{S}'},\mathfrak{b}_{\mathsf{S}''})}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{U}^n_{\mathsf{S}} \arrow[d, "\rho^n_{\mathsf{S}}"] \\
\mathcal{U}^n_{\mathfrak{b}_{\mathsf{S}'}} \arrow[r] & \mathcal{U}^n_{\mathfrak{b}_{\mathsf{S}}}
\end{tikzcd}$$

10.5. **Universal twists.** Let $\Phi\in\mathsf{Assoc}$ be a fixed associator. For any saturated subsemigroup $\mathsf{S}'\subseteq\mathsf{S}$, denote by $\Phi_{\mathsf{S}'}=\widetilde{\rho}^3_{\mathsf{S}'}(\Phi)$ the image of Φ in $\widehat{\mathfrak{U}}^3_{\mathsf{S}}$ under the map $\widetilde{\rho}^3_{\mathsf{S}'}:\mathfrak{U}^3_{\mathsf{DY}}\to\mathfrak{U}^3_{\mathsf{S}}$.

Let $J^{\mathrm{rel}}\in\widehat{\mathfrak{U}}^2_{\mathsf{PDY}}$ be the universal relative twist constructed in [1] (see Theorem 7.6). For any $\mathsf{S}''\subseteq\mathsf{S}'\subseteq\mathsf{S}$, set $J_{(\mathsf{S}',\mathsf{S}'')}=\widetilde{\rho}^2_{(\mathsf{S}',\mathsf{S}'')}(J^{\mathrm{rel}})\in\widehat{\mathfrak{U}}^2_{\mathsf{S}}$. Then, $J_{(\mathsf{S}',\mathsf{S}'')}\in\widehat{\mathfrak{U}}^2_{\mathsf{S}',\mathsf{S}''}=(\widehat{\mathfrak{U}}^2_{\mathsf{S}'})^{[\mathfrak{b}_{\mathsf{S}''}]}$ and it satisfies the relative twist equation

$$(\Phi_{\mathsf{S}'})_{J_{(\mathsf{S}',\mathsf{S}'')}}=\Phi_{\mathsf{S}''}\tag{10.2}$$

Theorem.

(1) *If* $J_1,J_2\in\widehat{\mathfrak{U}}^2_{\mathsf{S}',\mathsf{S}''}$*, are solutions of* (10.2)*, with* $(J_i)_0=1$*, there is a gauge transformation* $u\in\widehat{\mathfrak{U}}^{\times}_{\mathsf{S}',\mathsf{S}''}$*, with* $u_0=1$*, such that* $J_2=u_1\cdot u_2\cdot J_1\cdot u_{12}^{-1}$*.*

(2) *The gauge transformation* u *is unique.*

Proof. The proof of (1) follows verbatim that of Theorem 7.7. (2) Assume that $u\in\widehat{\mathfrak{U}}^{\times}_{\mathsf{S}',\mathsf{S}''}$ is such that

$$u_1\cdot u_2\cdot J=J\cdot u_{12}$$

and $u=1 \bmod (\mathfrak{U}_{\mathsf{S}',\mathsf{S}''})_{\geqslant n}$. Let $v\in(\mathfrak{U}_{\mathsf{S}',\mathsf{S}''})_n$ such that $u=1+v \bmod (\mathfrak{U}_{\mathsf{S}',\mathsf{S}''})_{\geqslant n+1}$. Taking the component of degree $n+1$ in the above equation yields

$$d_H(v)=v_2-v_{12}+v_1=0$$

which by Theorem 9.6 implies $v=0$. $\square$

10.6. **Orthogonal semigroups.** By definition, two subsemigroups $\mathsf{S}', \mathsf{S}'' \subseteq \mathsf{S}$ are *orthogonal* if $(\mathsf{S}' \times \mathsf{S}'') \cap \mathsf{S}^{(2)} = \emptyset$.

Proposition. *Let $\mathsf{S}', \mathsf{S}'' \subseteq \mathsf{S}$ be orthogonal saturated subsemigroups.*

(1) *In $\underline{\mathsf{DY}}^n_{\mathsf{S}}$, the action and coaction of $[\mathfrak{b}_{\mathsf{S}'}]$ on each $[\mathsf{V}_k]$ commute with those of $[\mathfrak{b}_{\mathsf{S}''}]$.*
(2) *Every element in $\mathfrak{U}^n_{\mathsf{S}'}$ commutes with the action and coaction of $[\mathfrak{b}_{\mathsf{S}''}]$ on any $[\mathsf{V}_k]$. In particular, every element in $\mathfrak{U}^n_{\mathsf{S}'}$ is $[\mathfrak{b}_{\mathsf{S}''}]$–invariant.*
(3) *In $\mathfrak{U}^n_{\mathsf{S}}$, $[\mathfrak{U}^n_{\mathsf{S}'}, \mathfrak{U}^n_{\mathsf{S}''}] = 0$.*
(4) *In $\mathfrak{U}^n_{\mathsf{S}}$, $\mathfrak{U}^n_{\mathsf{S}' \sqcup \mathsf{S}''} = \mathfrak{U}^n_{\mathsf{S}'} \cdot \mathfrak{U}^n_{\mathsf{S}''}$.*

Proof. (1) follows from the orthogonaliy of the subsemigroups, since it implies that, for any $\alpha \in \mathsf{S}', \beta \in \mathsf{S}''$, $\mu \circ \theta_\alpha \otimes \theta_\beta = 0 = \theta_\alpha \otimes \theta_\beta \circ \delta$. (2) and (3) are direct consequences of (1) since every element in $\mathfrak{U}^n_{\mathsf{S}'}$ (resp. $\mathfrak{U}^n_{\mathsf{S}''}$) is realised as a composition of actions and coaction of $[\mathfrak{b}_{\mathsf{S}'}]$ (resp. $[\mathfrak{b}_{\mathsf{S}''}]$). Finally, the same argument shows that $\mathfrak{U}^n_{\mathsf{S}' \sqcup \mathsf{S}''} = \mathfrak{U}^n_{\mathsf{S}'} \cdot \mathfrak{U}^n_{\mathsf{S}''}$. Namely, let $\underline{\alpha} \in (\mathsf{S}' \sqcup \mathsf{S}'')^N$ and define a partition I', I'' of $\{1, \ldots, N\}$ such that $\alpha_i \in \mathsf{S}'$, $i \in I'$, $\alpha_j \in \mathsf{S}''$, $j \in I''$. One has

$$r^{\underline{\alpha}, \sigma}_{\underline{N}, \underline{\widetilde{N}}} = r^{\underline{\alpha}', \sigma'}_{\underline{N}', \underline{\widetilde{N}}'} \cdot r^{\underline{\alpha}'', \sigma''}_{\underline{N}'', \underline{\widetilde{N}}''}$$

where $\underline{\alpha}' = (\alpha_i)_{i \in I'}$, $\underline{\alpha}'' = (\alpha_j)_{j \in I''}$, $\sigma' \in \mathfrak{S}_{|I'|}$, $\sigma'' \in \mathfrak{S}_{|I''|}$ are the restrictions of σ to I' and I'', respectively, and similarly for $\underline{N}', \underline{N}''$. Therefore, $r^{\underline{\alpha}', \sigma'}_{\underline{N}', \underline{\widetilde{N}}'} \in \mathfrak{U}^n_{\mathsf{S}'}$, $r^{\underline{\alpha}'', \sigma''}_{\underline{N}'', \underline{\widetilde{N}}''} \in \mathfrak{U}^n_{\mathsf{S}''}$, and (4) follows. $\square$

11. Diagrams and nested sets

We review in this section a number of combinatorial notions associated to a diagram D, in particular the definition of nested sets on D and their relative version, following [7, 27, 2].

11.1. **Nested sets on diagrams.** A *diagram* is an undirected graph D with no multiple edges or loops. A *subdiagram* $B \subseteq D$ is a full subgraph of D, that is, a graph consisting of a (possibly empty) subset of vertices of D, together with all edges of D joining any two elements of it.

Two subdiagrams $B_1, B_2 \subseteq D$ are *orthogonal* if they have no vertices in common, and no two vertices $i \in B_1$, $j \in B_2$ are joined by an edge in D. We denote by $B_1 \sqcup B_2$ the disjoint union of orthogonal subdiagrams. Two subdiagrams $B_1, B_2 \subseteq D$ are *compatible* if either one contains the other or they are orthogonal.

A *nested set* on D is a collection $\mathcal{H}$ of pairwise compatible, connected subdiagrams of D which contains the empty set and $\mathsf{conn}(D)$, where $\mathsf{conn}(D)$ denotes the set of connected components of D.

Let $\mathsf{Ns}(D)$ be the partial ordered set of nested sets on D, ordered by reverse inclusion. $\mathsf{Ns}(D)$ has a unique maximal element $\mathsf{conn}(D)$ and its minimal elements are the *maximal nested sets*. We denote the set of maximal nested sets on D by $\mathsf{Mns}(D)$. It is easy to see that the cardinality of any maximal nested set on D is equal to $|D| + 1$. Every nested set $\mathcal{H}$ on D is uniquely determined by a collection $\{\mathcal{H}_i\}_{i=1}^r$ of nested sets on the connected components D_i of D. We therefore obtain canonical identifications

$$\mathsf{Ns}(D) = \prod_{i=1}^r \mathsf{Ns}(D_i) \qquad \text{and} \qquad \mathsf{Mns}(D) = \prod_{i=1}^r \mathsf{Mns}(D_i).$$

11.2. **Relative nested sets.** If $B' \subseteq B \subseteq D$ are two subdiagrams of D, a nested set on B *relative* to B' is a collection of subdiagrams of B, containing $\mathsf{conn}(B)$ and $\mathsf{conn}(B')$, in which every element is compatible with, but not properly contained in any of the connected components of B'. We denote by $\mathsf{Ns}(B, B')$ and $\mathsf{Mns}(B, B')$, respectively, the collections of nested sets and maximal nested sets on B relative to B'. In particular, $\mathsf{Ns}(B) = \mathsf{Ns}(B, \emptyset)$ and $\mathsf{Mns}(B) = \mathsf{Mns}(B, \emptyset)$. Relative nested sets are endowed with the following operations, which preserve maximal nested sets.

- **Vertical union.** For any $B'' \subseteq B' \subseteq B$, there is an embedding
$$\cup : \mathsf{Ns}(B, B') \times \mathsf{Ns}(B', B'') \to \mathsf{Ns}(B, B''), \tag{11.1}$$
given by the union of nested sets. Its image $\mathsf{Ns}_{B'}(B, B'') \subseteq \mathsf{Ns}(B, B'')$ is the collection of relative nested sets which contains $\mathsf{conn}(B')$.
- **Vertical decomposition.** Let $B'' \subseteq B$ and $\mathcal{H} \in \mathsf{Ns}(B, B'')$. If $\mathsf{conn}(B') \subseteq \mathcal{H}$ and $B'' \subseteq B'$, $\mathcal{H}$ is in the image of (11.1). Therefore, there are uniquely defined nested sets $\mathcal{H}_{B''B'} \in \mathsf{Ns}(B', B'')$ and $\mathcal{H}_{B'B} \in \mathsf{Ns}(B, B')$ [17] such that
$$\mathcal{H} = \mathcal{H}_{B''B'} \cup \mathcal{H}_{B'B};$$
- **Orthogonal union.** For any $B = B_1 \sqcup B_2$ and $B' = B'_1 \sqcup B'_2$ with $B'_1 \subseteq B_1$, $B'_2 \subseteq B_2$, there is a bijection
$$\mathsf{Ns}(B_1, B'_1) \times \mathsf{Ns}(B_2, B'_2) \to \mathsf{Ns}(B, B'),$$
mapping $(\mathcal{H}_1, \mathcal{H}_2) \mapsto \mathcal{H}_1 \cup \mathcal{H}_2$.

12. Diagrammatic semigroups and Lie bialgebras

In this section, we introduce the notion of diagrammatic semigroup. The corresponding extension of $\underline{\mathsf{LBA}}$ allows to account for both the diagrammatic structure of the Borel subalgebra of a complex semisimple Lie algebra, as well as its root space decomposition.

12.1. **Lax D–algebras** [2, §3]**.** Let D be a diagram. A *lax D–algebra* is the datum of

- for any $B \subseteq D$, a k–algebra A_B
- for any $B' \subseteq B$, a homomorphism $i_{BB'} : A_{B'} \to A_B$

such that

- for any $B'' \subseteq B' \subseteq B$, $i_{BB'} \circ i_{B'B''} = i_{BB''}$
- for any $B = B' \sqcup B''$, with $B' \perp B''$, $m_B \circ i_{BB'} \otimes i_{BB''}$ is a morphism of algebras $A_{B'} \otimes A_{B''} \to A_B$, where m_B denotes the multiplication in A_B.

A *strict morphism* $A^1 \to A^2$ of lax D–algebras is a collection of algebra homomorphisms $\phi_B : A^1_B \to A^2_B$ labeled by the subdiagrams $B \subseteq D$ such that, for any $B' \subseteq B$, $i^2_{BB'} \circ \phi_{B'} = \phi_B \circ i^1_{BB'}$ as morphisms $A^1_{B'} \to A^2_B$.

[17] More precisely, for any $\mathcal{H} \in \mathsf{Mns}(B, B''')$ with $\mathsf{conn}(B'), \mathsf{conn}(B'') \in \mathcal{H}$ and $B''' \subseteq B'' \subseteq B'$, we set
$$\mathcal{H}_{B''B'} = \{C \in \mathcal{H} \mid C \subseteq B', C \subsetneq B''\}$$

12.2. **Diagrammatic Lie bialgebras** [2, §5]**.** A *diagrammatic Lie bialgebra* $\mathfrak{b}$ is the datum of

- a diagram D
- for any $B \subseteq D$, a Lie bialgebra $\mathfrak{b}_B$
- for any $B' \subseteq B$, a Lie bialgebra morphism $i_{BB'} : \mathfrak{b}_{B'} \to \mathfrak{b}_B$

such that

- for any B, $i_{BB} = \mathrm{id}_{\mathfrak{b}_B}$
- for any $B'' \subseteq B' \subseteq B$, $i_{BB'} \circ i_{B'B''} = i_{BB''}$
- for any $B = B' \sqcup B''$ with $B' \perp B''$, $i_{BB'} + i_{BB''} : \mathfrak{b}_{B'} \oplus \mathfrak{b}_{B''} \to \mathfrak{b}_B$ is an isomorphism of Lie bialgebras.

The above properties imply in particular that $\mathfrak{b}_\emptyset = 0$, and that $U\mathfrak{b}$ is a lax D–algebra, with $(U\mathfrak{b})_B = U\mathfrak{b}_B$.

12.3. **Split diagrammatic Lie bialgebras** [2, §5]**.** A *split pair* of Lie bialgebras $(\mathfrak{b}, \mathfrak{a})$ is the datum of two Lie bialgebras $\mathfrak{a}, \mathfrak{b}$, together with Lie bialgebra morphisms $i : \mathfrak{a} \to \mathfrak{b}$ and $p : \mathfrak{b} \to \mathfrak{a}$ such that $p \circ i = \mathrm{id}_\mathfrak{a}$. These give rise to an embedding $i \oplus p^t : \mathfrak{g}_\mathfrak{a} \hookrightarrow \mathfrak{g}_\mathfrak{b}$ of the corresponding doubles, which preserves the bracket and the inner product.

A diagrammatic Lie bialgebra $\mathfrak{b}$ is *split* if there are Lie bialgebra morphisms $p_{B'B} : \mathfrak{b}_B \to \mathfrak{b}_{B'}$ for any $B' \subseteq B$, such that $p_{B'B} \circ i_{BB'} = \mathrm{id}_{\mathfrak{b}_{B'}}$, and

- for any B, $p_{BB} = \mathrm{id}_{\mathfrak{b}_B}$
- for any $B'' \subseteq B' \subseteq B$, $p_{B''B'} \circ p_{B'B} = p_{B''B}$
- for any $B = B' \sqcup B''$ with $B' \perp B''$, $p_{B'B} \oplus p_{B''B} : \mathfrak{b}_B \to \mathfrak{b}_{B'} \oplus \mathfrak{b}_{B''}$ is an isomorphism of Lie bialgebras, and is the inverse of $i_{BB'} + i_{BB''}$.

12.4. **Example.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, with Borel and Cartan subalgebras $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$, Dynkin diagram D, Serre generators $\{e_i, f_i, \alpha_i^\vee\}_{i\in D}$, and standard Lie bialgebra structure (see 15.5). Then, $\mathfrak{g}$ is a diagrammatic Lie bialgebra, where, for any $B \subseteq D$, $\mathfrak{g}_B \subseteq \mathfrak{g}$ is the subalgebra generated by $\{e_i, f_i, \alpha_i^\vee\}_{i\in B}$.

$\mathfrak{b}$ is also a diagrammatic Lie bialgebra with subalgebras $\mathfrak{b}_B = \mathfrak{h}_B \oplus \mathfrak{n}_B$, where $\mathfrak{h}_B \subseteq \mathfrak{h}$ is the span of $\{\alpha_i^\vee\}_{i\in B}$ and $\mathfrak{n}_B$ is the nilpotent subalgebra generated by $\{e_i\}_{i\in B}$. Moreover, the diagrammatic structure on $\mathfrak{b}$ is split as follows. Let $\mathsf{R}_+ \subset \mathfrak{h}^*$ be the set of positive roots of $\mathfrak{g}$ relative to $\mathfrak{b}$ and, for any $B \subseteq D$, let $\mathsf{R}_{B,+} \subseteq \mathsf{R}_+$ be the subset of roots whose support lies in B. In particular, $\mathfrak{n}_B = \bigoplus_{\alpha\in\mathsf{R}_{B,+}} \mathfrak{g}_\alpha$. Then, for any $B' \subseteq B$, we have

$$\mathfrak{h}_B = \mathfrak{h}_{B'} \oplus \mathfrak{h}_{B'}^\perp \qquad \text{and} \qquad \mathfrak{n}_B = \mathfrak{n}_{B'} \oplus \mathfrak{n}_{B'}^\perp$$

where $\mathfrak{h}_{B'}^\perp = \{t \in \mathfrak{h}_B |\, \alpha_i(t) = 0, i \in B'\}$ and $\mathfrak{n}_{B'}^\perp = \bigoplus_{\alpha\in\mathsf{R}_{B,+}\setminus\mathsf{R}_{B',+}} \mathfrak{g}_\alpha$. The corresponding projections $p_{B'B} : \mathfrak{b}_B = \mathfrak{h}_B \oplus \mathfrak{n}_B \to \mathfrak{h}_{B'} \oplus \mathfrak{n}_{B'} = \mathfrak{b}_{B'}$ are Lie bialgebra morphisms and give rise to a split diagrammatic Lie bialgebra structure on $\mathfrak{b}$.

12.5. **Diagrammatic semigroups.** A *diagrammatic semigroup* is a pair $\mathbb{S} = (\mathsf{S}, D)$ where D is a diagram and S a semigroup endowed with a family of subsets $\mathsf{S}(B) \subseteq \mathsf{S}$ indexed by the subdiagrams of D, such that

- $\mathsf{S}(B)$ is a saturated subsemigroup in S
- $\mathsf{S}(B') \subseteq \mathsf{S}(B)$ for any $B' \subseteq B$
- for any $B' \perp B''$,

$$\mathsf{S}(B' \sqcup B'') = \mathsf{S}(B') \sqcup \mathsf{S}(B'') \qquad \text{and} \qquad (\mathsf{S}(B') \times \mathsf{S}(B'')) \cap (\mathsf{S})^{(2)} = \emptyset$$

It follows in particular that $\mathsf{S}(\emptyset)=\emptyset$. Moreover, if $\mathfrak{b}$ is an S–graded Lie bialgebra, and $B\subseteq D$ is a subdiagram, then

$$\mathfrak{b}_B\coloneqq\bigoplus_{\alpha\in\mathsf{S}(B)}\mathfrak{b}_\alpha$$

is a Lie subbialgebra of $\mathfrak{b}$. The following is straightforward.

Proposition. *Let $\mathbb{S}=(\mathsf{S},D)$ be a diagrammatic semigroup. Then, every $\mathbb{S}$–graded Lie bialgebra is a split diagrammatic Lie bialgebra.*

12.6. **Semisimple Lie algebras.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, with Borel subalgebra $\mathfrak{b}$. As pointed out in 8.6, $\mathfrak{b}$ is graded by $\mathsf{R}_0=\mathsf{R}_+\sqcup\{0\}$, where R_+ is the semigroup of positive roots of $\mathfrak{g}$ relative to $\mathfrak{b}$. However, the diagrammatic structure of $\mathfrak{b}$ given in 12.4 is not encoded by its R_0–grading via Proposition 12.5. Indeed, if $B\subsetneq D$, $\mathfrak{b}_B=\mathfrak{h}_B\oplus\mathfrak{n}_B$ does not correspond to a subset of R_0 since $\mathfrak{h}_B\subsetneq\mathfrak{h}$ is not a graded component of $\mathfrak{b}$.

Note that R_+ is a diagrammatic semigroup, with saturated subsemigroups $\mathsf{R}_{B,+}$ given by the set of roots with support in B, but this diagrammatic structure does not extend to R_0 since $\mathsf{R}_{B,+}$ is not saturated in R_0. Alternatively, one can consider the saturated subsemigroups $\mathsf{R}_0(B)=\mathsf{R}_{B,+}\sqcup\{0\}\subset\mathsf{R}_0$, but the latter only detect the Lie subbialgebras $\mathfrak{h}\oplus\mathfrak{n}_B\supset\mathfrak{b}_B$ and, correspondingly, do not satisfy the orthogonality property $(\mathsf{R}_0(B)\times\mathsf{R}_0(B'))\cap\mathsf{R}_0^{(2)}=\emptyset$ if $B\perp B'$.

The need to simultaneously account for the diagrammatic and the R_0–graded structure of $\mathfrak{b}$ motivates the construction in the following paragraph.

12.7. **Extensions of diagrammatic semigroups and PROPs.** Let $\mathbb{S}=(\mathsf{S},D)$ be a diagrammatic semigroup, and $\mathsf{S}_0=\mathsf{S}\sqcup\{0\}$ the semigroup which extends S with an element 0 such that $(0,0)\notin\mathsf{S}_0^{(2)}$ and $\alpha+0=\alpha$ for any $\alpha\in\mathsf{S}$. If $\alpha\in\mathsf{S}$ and $B\subseteq D$, we write $\alpha\perp B$ if $\alpha\in\mathsf{S}(B')$ for some $B'\perp B$.

Let $\underline{\mathsf{LBA}}_{\mathbb{S}}$ be the PROP generated by a module over $\underline{\mathsf{LBA}}_{\mathsf{S}_0}$,[18] and a family of idempotents $\theta_{0,B}:[1]\to[1]$, $B\subseteq D$, such that $\theta_{0,D}=\theta_0$,

$$\theta_{0,B'}\circ\theta_{0,B}=\theta_{0,B'}=\theta_{0,B}\circ\theta_{0,B'}\quad\text{for any } B'\subseteq B$$

$$\theta_{0,B'\sqcup B''}=\theta_{0,B'}+\theta_{0,B''}\quad\text{for any } B'\perp B''$$

and the following additional relations hold

$$\mu\circ\theta_{0,B}\otimes\theta_\alpha=\begin{cases}0 & \text{if } \alpha\perp B\\ \mu\circ\theta_0\otimes\theta_\alpha & \text{if } \alpha\in\mathsf{S}(B)\end{cases}$$

$$\theta_{0,B}\otimes\theta_\alpha\circ\delta=\begin{cases}0 & \text{if } \alpha\perp B\\ \theta_0\otimes\theta_\alpha\circ\delta & \text{if } \alpha\in\mathsf{S}(B)\end{cases}$$

The above relations imply that $\theta_{0,\emptyset}=0$, and that $\theta_{0,B'}\circ\theta_{0,B''}=0=\theta_{0,B''}\circ\theta_{0,B'}$ for any $B'\perp B''$ since if p,q are idempotents, $p+q$ is an idempotent if and only if $pq=0=qp$.

Proposition.

[18] That is, by a complete family of orthogonal idempotents $\theta_\alpha:[1]\to[1]$, $\alpha\in\mathsf{S}_0$, a bracket $\mu:[2]\to[1]$, and a cobracket $\delta:[1]\to[2]$, with the same relations described in 9.1.

(1) *For any* $B \subseteq D$*, set* $\theta_B = \theta_{0,B} + \sum_{\alpha \in \mathsf{S}(B)} \theta_\alpha$*. Then,* $\theta_B^2 = \theta_B$*,*

$$\theta_B \circ \mu = \mu \circ \theta_B \otimes \theta_B \qquad \textit{and} \qquad \delta \circ \theta_B = \theta_B \otimes \theta_B \circ \delta.$$

(2) *If* $\mathcal{N}$ *is a Karoubian,* k*–linear symmetric monoidal category, any module* $\mathfrak{b} \in \mathcal{N}$ *over* $\underline{\mathsf{LBA}}_{\mathbb{S}}$ *is a split diagrammatic Lie bialgebra with* $\mathfrak{b}_B = \theta_B(\mathfrak{b})$*,* $B \subseteq D$*.*

Proof. (1) The relations above imply in particular that $\theta_{0,B} \circ \theta_\alpha = 0 = \theta_\alpha \circ \theta_{0,B}$ for any $B \subseteq D$ and $\alpha \in \mathsf{S}$, so that $\theta_B^2 = \theta_B$. Set now $\theta'_B = \theta_0 + \sum_{\alpha \in \mathsf{S}(B)} \theta_\alpha$. Then, since $\mathsf{S}(B) \cup \{0\}$ is a saturated subsemigroup in S_0,

$$\mu \circ \theta_B \otimes \theta_B = \mu \circ \theta'_B \otimes \theta'_B = \theta'_B \circ \mu = \theta_B \circ \mu,$$

where the first equality follow from $\mu \circ \theta_0 \otimes \theta_0 = 0 = \mu \circ \theta_{0,B} \otimes \theta_{0,B}$ and $\mu \circ (\theta_0 - \theta_{0,B}) \otimes \theta_\alpha = 0$ if $\alpha \in \mathsf{S}(B)$, and the last one from $\theta_0 \circ \mu = 0 = \theta_{0,B} \circ \mu$. Similarly for δ. Moreover, for any $B' \perp B''$, $\mu \circ \theta_{B'} \otimes \theta_{B''} = 0 = \theta_{B'} \otimes \theta_{B''} \circ \delta$. It follows that $\theta_B(\mathfrak{b}), B \subseteq D$, are Lie bialgebras and define a split diagrammatic structure on $\mathfrak{b}$. (2) is a direct consequence of (1). $\square$

12.8. **Example.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and retain the notation of 12.6. Set $\mathbb{S} = (\mathsf{R}_+, D)$, where the diagrammatic structure on the semigroup R_+ is given by $\mathsf{R}_+(B) = \mathsf{R}_{B,+}$. Then, the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is a module over $\underline{\mathsf{LBA}}_{\mathbb{S}}$, where the idempotents $\theta_{0,B}$ correspond to the projections $\mathfrak{h} = \mathfrak{h}_B \oplus \mathfrak{h}_B^\perp \to \mathfrak{h}_B$. In particular, both the diagrammatic and R_0–graded structure of $\mathfrak{b}$ are codified by $\underline{\mathsf{LBA}}_{\mathbb{S}}$.

Henceforth, by abuse of terminology, we say that a Lie bialgebra is $\mathbb{S}$–*graded* if it is a $\underline{\mathsf{LBA}}_{\mathbb{S}}$–module.

12.9. **Colimit structure of** $\underline{\mathsf{LBA}}_{\mathbb{S}}$**.** The PROP $\underline{\mathsf{LBA}}_{\mathbb{S}}$ is not a semigroup extension of $\underline{\mathsf{LBA}}$ in the sense of 9.1, since the family of idempotents $\{\theta_\alpha, \theta_{0,B}\}$ is not labeled by a semigroup, and is neither complete nor orthogonal. Nevertheless, we show below that $\underline{\mathsf{LBA}}_{\mathbb{S}}$ is in fact a colimit of semigroup extensions of $\underline{\mathsf{LBA}}$.

We retain the notation from Section 11. For any $\mathcal{H} \in \mathsf{Ns}(D)$ and $B \in \mathcal{H}$, denote by $\underline{B}_{\mathcal{H}} \subset B$ the union of the maximal elements of $\mathcal{H}$ properly contained in B. Let $\mathsf{S}_{\mathcal{H}}$ be the semigroup with underlying set $\mathsf{S} \sqcup \{\zeta_B^{\mathcal{H}}\}_{B \in \mathcal{H}}$, which contains S as subsemigroup and is such that $\zeta_{B_1}^{\mathcal{H}} + \zeta_{B_2}^{\mathcal{H}}$ is undefined for any $B_1, B_2 \in \mathcal{H}$ and

$$\alpha + \zeta_B^{\mathcal{H}} = \begin{cases} \text{undefined} & \text{if } \alpha \perp B \text{ or } \alpha \in \mathsf{S}(\underline{B}_{\mathcal{H}}) \\ \alpha & \text{otherwise} \end{cases}$$

Proposition. *Set* $\underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}} = \underline{\mathsf{LBA}}_{\mathsf{S}_{\mathcal{H}}}$*.*

(1) *For any* $\mathcal{H}$*, there is a morphism* $h_{\mathcal{H}} : \underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}} \to \underline{\mathsf{LBA}}_{\mathbb{S}}$ *given by* $h_{\mathcal{H}}(\mu) = \mu$*,* $h_{\mathcal{H}}(\delta) = \delta$*,* $h_{\mathcal{H}}(\theta_\alpha) = \theta_\alpha$*,* $\alpha \in \mathsf{S}$*, and*

$$h_{\mathcal{H}}(\theta_{\zeta_B^{\mathcal{H}}}) = \theta_{0,B} - \theta_{0,\underline{B}_{\mathcal{H}}}$$

(2) *For any* $\mathcal{H}' \subseteq \mathcal{H}$*, there is a morphism* $h_{\mathcal{H}\mathcal{H}'} : \underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}'} \to \underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}}$ *given by* $h_{\mathcal{H}\mathcal{H}'}(\mu) = \mu$*,* $h_{\mathcal{H}\mathcal{H}'}(\delta) = \delta$*,* $h_{\mathcal{H}\mathcal{H}'}(\theta_\alpha) = \theta_\alpha$*,* $\alpha \in \mathsf{S}$*, and*

$$h_{\mathcal{H}\mathcal{H}'}(\theta_{\zeta_{B'}^{\mathcal{H}'}}) = \sum_B \theta_{\zeta_B^{\mathcal{H}}}$$

where the sum ranges over the $B \in \mathcal{H}$ *such that* $B \subseteq B'$*, and* $B \not\subseteq \underline{B'}_{\mathcal{H}'}$*.*

(3) *The following holds for any* $\mathcal{H}'' \subseteq \mathcal{H}' \subseteq \mathcal{H}$,

$$h_{\mathcal{H}} \circ h_{\mathcal{H}\mathcal{H}'} = h_{\mathcal{H}'} \qquad \text{and} \qquad h_{\mathcal{H}\mathcal{H}'} \circ h_{\mathcal{H}'\mathcal{H}''} = h_{\mathcal{H}\mathcal{H}''}$$

as morphisms $\underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}'} \to \underline{\mathsf{LBA}}_{\mathbb{S}}$ *and* $\underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}''} \to \underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}}$ *respectively.*

(4) *The* PROP $\underline{\mathsf{LBA}}_{\mathbb{S}}$ *is the colimit of* $(\underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}}, h_{\mathcal{H}\mathcal{H}'})$.

Proof. (1)–(3) are verified by direct inspection. (4) Let $\underline{\mathsf{P}}$ be a PROP endowed with a family of morphisms $p_{\mathcal{H}} : \underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}} \to \underline{\mathsf{P}}$ such that $p_{\mathcal{H}} \circ h_{\mathcal{H}\mathcal{H}'} = p_{\mathcal{H}'}$ for any $\mathcal{H}' \subseteq \mathcal{H}$. Then, one can check easily that there is a unique morphism p : $\underline{\mathsf{LBA}}_{\mathbb{S}} \to \underline{\mathsf{P}}$ such that $p \circ h_{\mathcal{H}=} p_{\mathcal{H}}$. Namely, p is determined by the assignment $p(\theta_{0,B}) = p_{\{B,D\}}(\theta_{\zeta_B^{\{B,D\}}})$. $\square$

12.10. **Universal Drinfeld–Yetter modules.** Fix henceforth a diagrammatic semigroup $\mathbb{S} = (\mathsf{S}, D)$. The category $\underline{\mathsf{DY}}^n_{\mathbb{S}}$, $n \geqslant 1$, is the colored PROP generated by $n+1$ objects, $[1]$ and $\{[\mathsf{V}_k]\}_{k=1}^n$, and morphisms

- $\theta_\alpha : [1] \to [1]$, $\alpha \in \mathsf{S}$, and $\theta_{0,B} : [1] \to [1]$, $B \subseteq D$
- $\mu : [2] \to [1]$, $\delta : [1] \to [2]$
- $\pi_k : [1] \otimes [\mathsf{V}_k] \to [\mathsf{V}_k]$, $\pi_k^* : [\mathsf{V}_k] \to [1] \otimes [\mathsf{V}_k]$

such that

- $([1], \theta_\alpha, \theta_{0,B}, \mu, \delta)$ is an $\underline{\mathsf{LBA}}_{\mathbb{S}}$–module in $\underline{\mathsf{DY}}^n_{\mathbb{S}}$
- every $([\mathsf{V}_k], \pi_k, \pi_k^*)$ is a Drinfeld–Yetter module over $[1]$

12.11. **Universal algebras.** We set $\mathfrak{U}^n_{\mathbb{S}} = \mathsf{End}_{\underline{\mathsf{DY}}^n_{\mathbb{S}}}([\mathsf{V}_1] \otimes [\mathsf{V}_2] \otimes \cdots \otimes [\mathsf{V}_n])$. For any $\mathbb{S}$–graded Lie bialgebra $\mathfrak{b}$ and n–tuple $\{V_k, \pi_k, \pi_k^*\}_{k=1}^n$ of Drinfeld–Yetter $\mathfrak{b}$–modules, there is a canonical realisation functor

$$\mathcal{G}_{(\mathfrak{b}, V_1, \dots, V_n)} : \underline{\mathsf{DY}}^n_{\mathbb{S}} \longrightarrow \mathrm{Vect}_{\mathsf{k}}$$

sending $[1] \mapsto \mathfrak{b}$, and $[\mathsf{V}_k] \mapsto V_k$. As usual, the functors $\mathcal{G}_{(\mathfrak{b}, V_1, \dots, V_n)}$ induce an algebra homomorphism $\rho^n_{\mathsf{S},\mathfrak{b}} : \mathfrak{U}^n_{\mathbb{S}} \to \mathcal{U}^n_{\mathfrak{b}}$, where $\mathcal{U}^n_{\mathfrak{b}} = \mathsf{End}\left(\mathsf{f}^{\boxtimes n}\right)$ and $\mathsf{f} \colon \mathsf{DY}_{\mathfrak{b}} \to \mathsf{Vect}_{\mathsf{k}}$ is the forgetful functor.

12.12. **Colimit structures.** As in the case of $\underline{\mathsf{LBA}}_{\mathbb{S}}$, $\underline{\mathsf{DY}}^n_{\mathbb{S}}$ is a colimit of semigroup extensions of $\underline{\mathsf{DY}}^n$. Namely, let $\mathcal{H} \in \mathsf{Ns}(D)$ and set $\underline{\mathsf{DY}}^n_{\mathsf{S},\mathcal{H}} = \underline{\mathsf{DY}}^n_{\mathsf{S}_{\mathcal{H}}}$ and $\mathfrak{U}^n_{\mathsf{S},\mathcal{H}} = \mathfrak{U}^n_{\mathsf{S}_{\mathcal{H}}}$. The morphism $h_{\mathcal{H}} : \underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}} \to \underline{\mathsf{LBA}}_{\mathbb{S}}$ (resp. $h_{\mathcal{H}\mathcal{H}'} : \underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}'} \to \underline{\mathsf{LBA}}_{\mathsf{S},\mathcal{H}}$, $\mathcal{H}' \subseteq \mathcal{H}$) extend immediately to a morphism of PROPs $h^n_{\mathcal{H}} : \underline{\mathsf{DY}}^n_{\mathsf{S},\mathcal{H}} \to \underline{\mathsf{DY}}^n_{\mathbb{S}}$ (resp. $h^n_{\mathcal{H}\mathcal{H}'} : \underline{\mathsf{DY}}^n_{\mathsf{S},\mathcal{H}'} \to \underline{\mathsf{DY}}^n_{\mathsf{S},\mathcal{H}}$) and then to a morphism of algebras $h^n_{\mathcal{H}\mathcal{H}'} : \mathfrak{U}^n_{\mathsf{S},\mathcal{H}'} \to \mathfrak{U}^n_{\mathsf{S},\mathcal{H}}$, compatibly with the inclusion of nested sets. The following corollary of Proposition 12.9 is immediate.

Corollary.

(1) *The* PROP $\underline{\mathsf{DY}}^n_{\mathbb{S}}$ *is the colimit of the system* $(\underline{\mathsf{DY}}^n_{\mathsf{S},\mathcal{H}}, h^n_{\mathcal{H}\mathcal{H}'})$.

(2) *The algebra* $\mathfrak{U}^n_{\mathbb{S}}$ *is the colimit of the system* $(\mathfrak{U}^n_{\mathsf{S},\mathcal{H}}, h^n_{\mathcal{H}\mathcal{H}'})$.

12.13. **Hochschild cohomology.** The cosimplicial structure and the Hochschild differential on $\mathfrak{U}^\bullet_{\mathbb{S}}$ are defined as in 5.13. Relying on the colimit structure described above, the results of Section 9, in particular the PBW theorem (Thm. 9.5) and computation of the Hochschild cohomology (Thm. 9.6), extend immediately to the case of a diagrammatic semigroup. More precisely, let $\mathcal{I}_{\mathbb{S}} \subseteq \underline{\mathsf{LBA}}_{\mathbb{S}}([1],[1])$ be the subset containing the idempotents θ_α, $\alpha \in \mathsf{S}$, and all iterated products of $\theta_{0,B}$, corresponding to connected subdiagrams $B \subseteq D$. For any $\underline{\alpha} \in \mathcal{I}^N_{\mathbb{S}}$, the Lie algebras $\mathcal{L}_{N,\underline{\alpha}}$ are defined as in 9.3. We have the following

Theorem. *The Hochschild cohomology of $\mathfrak{U}_{\mathbb{S}}^{\bullet}$ is described as follows:*

$$\begin{aligned} H^i(\mathfrak{U}_{\mathbb{S}}^{\bullet}, d_H) &\cong \bigoplus_{j=0}^{i} \underline{\mathsf{LBA}}_{\mathbb{S}}(\wedge^j[1], \wedge^{i-j}[1]) \\ &\cong \bigoplus_{N\geqslant 0}\bigoplus_{j=0}^{i}\left(\prod_{\underline{\alpha}\in\mathcal{I}_{\mathbb{S}}^N}\left(\wedge^j\mathcal{L}_{N,\underline{\alpha}}\right)_{\delta_N}\otimes\left(\wedge^{i-j}\mathcal{L}_{N,\underline{\alpha}}\right)_{\delta_N}\right)_{\mathfrak{S}_N}\end{aligned}$$

In particular, $H^0(\mathfrak{U}_{\mathbb{S}}^{\bullet}, d_H) = \mathsf{k}$ and $H^1(\mathfrak{U}_{\mathbb{S}}^{\bullet}, d_H) = 0$.

12.14. **Diagrammatic subalgebras.** For any $B \subseteq D$, set $\mathbb{S}(B) = (\mathsf{S}(B), B)$. Let $\mathfrak{U}_{\mathbb{S},B}^n := \mathfrak{U}_{\mathbb{S}(B)}^n$ be the universal algebra in the PROP $\underline{\mathsf{DY}}_{\mathbb{S},B}^n := \underline{\mathsf{DY}}_{\mathbb{S}(B)}^n$. For any $B \subseteq B'$, there is a canonical realisation functor

$$\mathcal{G}_{\theta_B[1],[\mathsf{V}_1],\dots,[\mathsf{V}_n]} : \underline{\mathsf{DY}}_B^n \to \underline{\mathsf{DY}}_{B'}^n$$

which sends the object $[1]_B$ in $\underline{\mathsf{DY}}_B^n$ to the Lie bialgebra object $\theta_B[1]_{B'} = ([1]_{B'}, \theta_B)$ in $\underline{\mathsf{DY}}_{\mathbb{S},B'}^n$ and induces a homomorphism $\mathsf{f}_{B'B} : \mathfrak{U}_{\mathbb{S},B}^n \to \mathfrak{U}_{\mathbb{S},B'}^n$.

For any $\mathbb{S}$–graded Lie bialgebra $\mathfrak{b}$ with diagrammatic Lie subbialgebras $\mathfrak{b}_B = \theta_B(\mathfrak{b})$ (cf. 12.7), we set $\mathcal{U}_{\mathfrak{b},B}^n = \mathsf{End}\left(\mathsf{f}_B^{\boxtimes n}\right)$, where $\mathsf{f}_B : \mathsf{DY}_{\mathfrak{b}_B} \to \mathsf{Vect}_{\mathsf{k}}$ is the forgetful functor, and we define $\rho_{\mathfrak{b},B}^n : \mathfrak{U}_{\mathbb{S},B}^n \to \mathcal{U}_{\mathfrak{b},B}^n$ as in 12.10.

Proposition.

(1) *The subalgebras $\{\mathfrak{U}_{\mathbb{S},B}^n\}_{B\subseteq D}$ define a lax D–algebra structure on $\mathfrak{U}_{\mathbb{S}}^n$.*
(2) *The subalgebras $\{\mathcal{U}_{\mathfrak{b},B}^n\}_{B\subseteq D}$ defines a lax D–algebra structure on $\mathcal{U}_{\mathfrak{b}}^n$.*
(3) *The collection of homomorphisms $\{\rho_{\mathfrak{b},B}^n : \mathfrak{U}_{\mathbb{S},B}^n \to \mathcal{U}_{\mathfrak{b},B}^n\}_{B\subseteq D}$, defines a strict morphism of D–algebras $\rho_{\mathfrak{b}}^n : \mathfrak{U}_{\mathbb{S}}^n \to \mathcal{U}_{\mathfrak{b}}^n$.*

PROOF. (1) follows from Proposition 10.6 (3), since the algebras $\mathfrak{U}_{\mathbb{S},B}^n$ are colimits of semigroup universal algebras. (2) is a consequence of the diagrammatic structure of the $\mathbb{S}$–graded Lie bialgebra $\mathfrak{b}$, and (3) is proved by direct inspection. □

An analogue result holds for the grading completions of $\mathfrak{U}_{\mathbb{S}}^n$ and $\mathcal{U}_{\mathfrak{b}}^n$, which are defined as in 7.1 and denoted $\widehat{\mathfrak{U}}_{\mathbb{S}}^n$ and $\widehat{\mathcal{U}}_{\mathfrak{b}}^n$.

12.15. **Uniqueness of twists in $\mathfrak{U}_{\mathbb{S}}^{\bullet}$.** For any subdiagrams $B' \subseteq B \subseteq D$, we denote by $\widehat{\mathfrak{U}}_{\mathbb{S},B,B'}^n$ the elements in $\widehat{\mathfrak{U}}_{\mathbb{S},B}^n$ which are invariant with respect to the action and coaction of $[\mathfrak{b}_{B'}] = ([1], \theta_{\mathsf{S}(B')})$. Relying on the description of the cohomology of $\mathfrak{U}_{\mathbb{S}}^n$ given by Theorem 12.13, we proceed as in Section 10 and prove the analogue of Theorem 10.5.

Theorem.

(1) *If $J_1, J_2 \in \widehat{\mathfrak{U}}_{\mathbb{S},B,B'}^2$ are solutions of the relative twist equation $(\Phi_B)_{J_i} = \Phi_{B'}$, with $(J_i)_0 = 1$, there is a gauge transformation $u \in \widehat{\mathfrak{U}}_{\mathbb{S},B,B'}^{\times}$, with $u_0 = 1$, such that $J_2 = u_1 \cdot u_2 \cdot J_1 \cdot u_{12}^{-1}$.*
(2) *The gauge transformation u is unique.*

13. Universal braided pre–Coxeter structures

Let $\mathbb{S} = (\mathsf{S}, D)$ be a diagrammatic semigroup, and $\mathfrak{U}_{\mathbb{S}}^n$ the universal algebras arising from the PROP $\underline{\mathsf{LBA}}_{\mathbb{S}}$. We define in this section the notion of braided pre–Coxeter structure on $\mathfrak{U}_{\mathbb{S}}^{\bullet}$, and prove its rigidity. We will prove in Section 14 that such structures give rise to braided Coxeter categories, as defined in [2].

13.1. **Pre–Coxeter structures on $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$.** Recall that, for any $B'\subseteq B\subseteq D$, $\widehat{\mathfrak{U}}^n_{\mathbb{S},B,B'}$ denotes the elements in $\widehat{\mathfrak{U}}^n_{\mathbb{S},B}$ which are invariant with respect to the action and coaction of $[\mathfrak{b}_{B'}]=([1],\theta_{\mathsf{S}(B')})$.

Definition. A *braided pre–Coxeter structure* $(\Phi_B, J_{\mathcal{F}}, \Upsilon_{\mathcal{F}\mathcal{G}})$ on $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$ consists of the following data.

(1) For any $B\subseteq D$, an associator $\Phi_B\in\widehat{\mathfrak{U}}^3_{\mathbb{S},B,B}$ (cf. Definition 7.3), satisfying the following orthogonal factorisation property. For any $B=B_1\sqcup B_2$,
$$\Phi_B=\Phi_{B_1}\cdot\Phi_{B_2}$$
We set $R_B=\exp(\Omega_B/2)\in\widehat{\mathfrak{U}}^2_{\mathbb{S},B,B}$, and note that $R_{B_1\sqcup B_2}=R_{B_1}\cdot R_{B_2}$.

(2) For any $B'\subsetneq B\subseteq D$, and maximal nested set $\mathcal{F}\in\mathsf{Mns}(B,B')$, a *relative twist* $J_{\mathcal{F}}\in\widehat{\mathfrak{U}}^2_{\mathbb{S},B,B'}$, that is an (invertible) element such that $(J_{\mathcal{F}})_0=1$ and $\varepsilon^1_2(J_{\mathcal{F}})=1=\varepsilon^2_2(J_{\mathcal{F}})$, where $\varepsilon^1_2,\varepsilon^2_2:\widehat{\mathfrak{U}}^2_{\mathbb{S},B}\to\widehat{\mathfrak{U}}_{\mathbb{S},B}$ are the degeneration homomorphisms, which is a solution of the relative twist equation
$$(\Phi_B)_{J_{\mathcal{F}}}=\Phi_{B'}$$
where $\Phi_J:=J^{23}J^{1,23}\Phi(J^{12,3})^{-1}(J^{12})^{-1}$ (cf. Section 7.6 and equations (7.1)–(7.2)). Moreover, the twists $J_{\mathcal{F}}$ satisfy the following factorisation properties.

- **Vertical factorisation.** For any $B''\subseteq B'\subseteq B$, $\mathcal{F}\in\mathsf{Mns}(B,B')$ and $\mathcal{F}'\in\mathsf{Mns}(B',B'')$
$$J_{\mathcal{F}\cup\mathcal{F}'}=J_{\mathcal{F}}\cdot J_{\mathcal{F}'}$$
In particular, $J_{\mathcal{F}}=1$ if $B'=B$ and $\mathcal{F}$ is the unique element in $\mathsf{Mns}(B,B)$.
- **Orthogonal factorisation.** For any $B=B_1\sqcup B_2$ and $B'=B'_1\sqcup B'_2$, with $B'_1\subseteq B_1$ and $B'_2\subseteq B_2$, and any orthogonal pair $\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2)\in\mathsf{Mns}(B_1,B'_1)\times\mathsf{Mns}(B_2,B'_2)=\mathsf{Mns}(B,B')$
$$J_{\mathcal{F}}=J_{\mathcal{F}_1}\cdot J_{\mathcal{F}_2}$$

(3) For any $B'\subseteq B\subseteq D$, and $\mathcal{F},\mathcal{G}\in\mathsf{Mns}(B,B')$, an invertible element $\Upsilon_{\mathcal{G}\mathcal{F}}\in\widehat{\mathfrak{U}}_{\mathbb{S},B,B'}$, henceforth referred to as a *De Concini–Procesi associator*, such that $(\Upsilon_{\mathcal{G}\mathcal{F}})_0=1$, $\varepsilon(\Upsilon_{\mathcal{G}\mathcal{F}})=1$, and
$$J_{\mathcal{G}}=(\Upsilon_{\mathcal{G}\mathcal{F}})_1\cdot(\Upsilon_{\mathcal{G}\mathcal{F}})_2\cdot J_{\mathcal{F}}\cdot(\Upsilon_{\mathcal{G}\mathcal{F}})^{-1}_{12}$$
The associators $\Upsilon_{\mathcal{G}\mathcal{F}}$ satisfy the following properties.

- **Transitivity.** For any $\mathcal{F},\mathcal{G},\mathcal{H}\in\mathsf{Mns}(B,B')$,
$$\Upsilon_{\mathcal{H}\mathcal{F}}=\Upsilon_{\mathcal{H}\mathcal{G}}\cdot\Upsilon_{\mathcal{G}\mathcal{F}}$$
In particular, $\Upsilon_{\mathcal{F}\mathcal{F}}=1$ and $\Upsilon_{\mathcal{G}\mathcal{F}}=\Upsilon^{-1}_{\mathcal{F}\mathcal{G}}$.
- **Vertical factorisation.** For any $B''\subseteq B'\subseteq B$, $\mathcal{F},\mathcal{G}\in\mathsf{Mns}(B,B')$ and $\mathcal{F}',\mathcal{G}'\in\mathsf{Mns}(B',B'')$,
$$\Upsilon_{(\mathcal{G}\cup\mathcal{G}')(\mathcal{F}\cup\mathcal{F}')}=\Upsilon_{\mathcal{G}\mathcal{F}}\cdot\Upsilon_{\mathcal{G}'\mathcal{F}'}$$
- **Orthogonal factorisation.** For any $B=B_1\sqcup B_2$ and $B'=B'_1\sqcup B'_2$, with $B'_1\subseteq B_1$ and $B'_2\subseteq B_2$, and orthogonal pairs $\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2)$ and $\mathcal{G}=(\mathcal{G}_1,\mathcal{G}_2)$ in $\mathsf{Mns}(B_1,B'_1)\times\mathsf{Mns}(B_2,B'_2)=\mathsf{Mns}(B,B')$
$$\Upsilon_{\mathcal{G}\mathcal{F}}=\Upsilon_{\mathcal{G}_1\mathcal{F}_1}\cdot\Upsilon_{\mathcal{G}_2\mathcal{F}_2}$$

13.2. **Twisting of braided pre–Coxeter structures on $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$.**

Definition.

(1) A *twist* in $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$ is a pair (u, F) where

(a) $u = \{u_{\mathcal{F}}\}$ is a collection of invertible elements in $\widehat{\mathfrak{U}}_{\mathbb{S},B',B}$, indexed by pairs of subdiagrams $B' \subseteq B$ and a maximal nested set $\mathcal{F} \in \mathsf{Mns}(B, B')$, which satisfy $\varepsilon(u_{\mathcal{F}}) = 1$, and the following factorisation properties.

- **Vertical factorisation.** For any $B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$, and $\mathcal{F}' \in \mathsf{Mns}(B', B'')$,

$$u_{\mathcal{F}\cup\mathcal{F}'} = u_{\mathcal{F}} \cdot u_{\mathcal{F}'} \tag{13.1}$$

- **Orthogonal factorisation.** For any $B = B_1 \sqcup B_2$ and $B' = B'_1 \sqcup B'_2$, with $B'_1 \subseteq B_1$ and $B'_2 \subseteq B_2$, and orthogonal pair $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ in $\mathsf{Mns}(B_1, B'_1) \times \mathsf{Mns}(B_2, B'_2) = \mathsf{Mns}(B, B')$,

$$u_{\mathcal{F}} = u_{\mathcal{F}_1} \cdot u_{\mathcal{F}_2} \tag{13.2}$$

(b) $F = \{F_B\}$ is a collection of invertible elements of $\widehat{\mathfrak{U}}^{2}_{\mathbb{S},B,B}$, indexed by subdiagrams $B \subseteq D$, which satisfy $\varepsilon^1_2(F_B) = 1 = \varepsilon^2_2(F_B)$, are symmetric,*i.e.*, $(F_B)_{21} = F_B$ (cf. 7.2), $d_H(F_B)_1 = 0$, and, for any $B = B_1 \sqcup B_2$,

$$F_B = F_{B_1} \cdot F_{B_2}$$

(2) The *twisting* of a braided pre–Coxeter structure $\mathcal{C} = (\Phi_B, J_{\mathcal{F}}, \Upsilon_{\mathcal{F}\mathcal{G}})$ by a twist (u, F) is the braided pre–Coxeter structure

$$\mathcal{C}_{(u,F)} = ((\Phi_B)_{F_B}, (J_{\mathcal{F}})_{(u,F)}, (\Upsilon_{\mathcal{F}\mathcal{G}})_u)$$

given by

$$\begin{aligned}
(\Phi_B)_{F_B} &= (F_B)_{23} \cdot (F_B)_{1,23} \cdot \Phi_B \cdot (F_B)^{-1}_{12,3} \cdot (F_B)^{-1}_{12} \\
(J_{\mathcal{F}})_{(u,F)} &= F_{B'}(u_{\mathcal{F}})_1 \cdot (u_{\mathcal{F}})_2 \cdot J_{\mathcal{F}} \cdot (u_{\mathcal{F}})^{-1}_{12} \cdot F_B^{-1} \\
(\Upsilon_{\mathcal{F}\mathcal{G}})_u &= u_{\mathcal{F}} \cdot \Upsilon_{\mathcal{F}\mathcal{G}} \cdot u_{\mathcal{G}}^{-1}
\end{aligned}$$

Remark. The twisting of a braided pre–Coxeter structure does not affect the R–matrix $R_B = \exp(\Omega_B/2)$. Specifically, set

$$(R_B)_{F_B} = (F_B)_{21} R_B F_B^{-1}$$

Since $2\Omega_B = \Delta(\kappa_B) - ((\kappa_B)_1 + (\kappa_B)_2)$, we have

$$\begin{aligned}
(R_B)_{F_B} &= F_B^{21} \exp(\Delta(\kappa_B)/2) \exp(-((\kappa_B)_1 + (\kappa_B)_2)/2) F_B^{-1} \\
&= \exp(\Delta(\kappa_B)/2) F_B^{21} \exp(-((\kappa_B)_1 + (\kappa_B)_2)/2) F_B^{-1} \\
&= \exp(\Delta(\kappa_B)/2) \exp(-((\kappa_B)_1 + (\kappa_B)_2)/2) = R_B
\end{aligned}$$

where the first identity follows from the invariance of F_B, and the second one from the fact that $(\kappa_B)_1 + (\kappa_B)_2$ is central in $\mathfrak{U}^2_{\mathbb{S},B}$ (Prop. 9.8) and $F_B^{21} = F_B$ by assumption.

Finally, we observe that the conditions $(R_B)_{F_B} = R_B$ and $d_H(F_B)_1 = 0$ guarantee that the 2–jet of the associator is preserved, *i.e.*, $((\Phi_B)_{F_B})_1 = 0$, and therefore $((\Phi_B)_{F_B})_2 = [\Omega_{B,12}, \Omega_{B,23}]/24$ by [9, Prop. 3.1].

13.3. **Gauging of twists transformation.**

Definition.

(1) A *gauge* is a collection $a = \{a_B\}$ of invertible elements $a_B \in \widehat{\mathfrak{U}}_{\mathbb{S},B,B}$ indexed by subdiagrams $B \subseteq D$ and satisfying, for any $B = B_1 \sqcup B_2$,

$$a_B = a_{B_1} \cdot a_{B_2}$$

(2) The *gauging* of a twist (u, F) by a is the twist (u_a, F_a) given by

$$(u_{\mathcal{F}})_a = a_{B'} \cdot u_{\mathcal{F}} \cdot a_B^{-1}$$
$$(F_B)_a = (a_B)_1 (a_B)_2 \cdot F_B \cdot (a_B)_{12}^{-1}$$

Remark. It is easy to see that if (u, F) is a twist, and a a gauge, the twist of a braided pre–Coxeter structure on $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$ by (u, F) is the same as that by (u_a, F_a).

13.4. **Uniqueness of braided pre–Coxeter structures on $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$.**

Theorem. *Let $\mathcal{C}_k = (\Phi_B^{(k)}, J_{\mathcal{F}}^{(k)}, \Upsilon_{\mathcal{F}\mathcal{G}}^{(k)})$, $k = 1, 2$, be two braided pre–Coxeter structures on $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$. Then*

(1) *There exists a twist (u, F) such that $u_0 = 1, F_0 = 1$, and*

$$\mathcal{C}_2 = (\mathcal{C}_1)_{(u,F)}$$

(2) *The twist (u, F) is unique up to a unique gauge a.*

Proof. We first match the associators. The proof of Drinfeld's uniqueness theorem [9, Prop. 3.12] is easily adapted to $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$. Namely, given $\Phi_B^{(1)}, \Phi_B^{(2)} \in \widehat{\mathfrak{U}}^{3}_{\mathbb{S},B}$, there is a symmetric, invariant twist $F_B \in \widehat{\mathfrak{U}}^{2}_{\mathbb{S},B}$ such that $(\Phi_B^{(1)})_{F_B} = \Phi_B^{(2)}$. In particular, $d_H(F_B)_1 = 0$. F_B is uniquely defined up to multiplication with an element of the form $(a_B)_1^{-1}(a_B)_2^{-1}(a_B)_{12}$, where a_B belongs to the center of $\widehat{\mathfrak{U}}_{\mathbb{S},B}$ and such that $(a_B)_0 = 1$. Further, $(R_B)_{F_B} = R_B$, since $R_B = \exp(\kappa_B/2)_{12} \cdot \exp(-((\kappa_B)_1 + (\kappa_B)_2)/2)$, as we explain in 13.2.

We may therefore assume that $\mathcal{C}_k = (\Phi_B, J_{\mathcal{F}}^{(k)}, \Upsilon_{\mathcal{F}\mathcal{G}}^{(k)})$. We now match the twists. By 12.15, there exists, for any $\mathcal{F} \in \mathsf{Mns}(B, B')$, an invertible element $u_{\mathcal{F}} \in \widehat{\mathfrak{U}}_{\mathbb{S},B,B'}$ satisfying

$$J_{\mathcal{F}}^{(2)} = (u_{\mathcal{F}})_1 (u_{\mathcal{F}})_2 J_{\mathcal{F}}^{(1)} (u_{\mathcal{F}})_{12}^{-1}$$

Moreover, it follows by Theorem 12.15 that the gauge transformation is unique, and therefore that $u = \{u_{\mathcal{F}}\}$ satisfies the factorisation properties (13.1) and (13.2). Therefore we can assume $\mathcal{C}_k = (\Phi_B, J_{\mathcal{F}}, \Upsilon_{\mathcal{F}\mathcal{G}}^{(k)})$ and

$$(\Upsilon_{\mathcal{F}\mathcal{G}}^{(k)})_1 \cdot (\Upsilon_{\mathcal{F}\mathcal{G}}^{(k)})_2 \cdot J_{\mathcal{G}} \cdot (\Upsilon_{\mathcal{F}\mathcal{G}}^{(k)})_{12}^{-1} = J_{\mathcal{F}}$$

fro $k = 1, 2$. Again by uniqueness, it follows that $\Upsilon_{\mathcal{F}\mathcal{G}}^{(1)} = \Upsilon_{\mathcal{F}\mathcal{G}}^{(2)}$. $\mathcal{C}_2$ is therefore a twist of $\mathcal{C}_1$, and the twist is uniquely defined up to a unique gauge. $\square$

14. Braided Coxeter categories

In this section, we review the definition of braided (pre–)Coxeter categories given in [2]. We then show that if $\mathbb{S}$ is a diagrammatic semigroup, and $\mathfrak{b}$ an $\mathbb{S}$–graded Lie bialgebra, a braided pre–Coxeter structure on the universal algebras $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$ endows Drinfeld–Yetter modules over the diagrammatic subalgebras of $\mathfrak{b}$ with the structure of a braided pre–Coxeter category.

14.1. Let D be a diagram. A *braided pre–Coxeter category* $\mathcal{C}$ of type D consists of the following data

- **Diagrammatic categories.** For any subdiagram $B \subseteq D$, a braided tensor category $\mathcal{C}_B$.
- **Restriction functors.** For any inclusion $B' \subseteq B$ and relative maximal nested set $\mathcal{F} \in \mathsf{Mns}(B, B')$, a tensor functor $F_\mathcal{F} : \mathcal{C}_B \to \mathcal{C}_{B'}$.
- **De Concini–Procesi associators.** For any inclusion $B' \subseteq B$ and pair of relative maximal nested sets $\mathcal{F}, G \in \mathsf{Mns}(B, B')$, an isomorphism of tensor functors $\Upsilon_{\mathcal{G}\mathcal{F}} : F_\mathcal{F} \Rightarrow F_\mathcal{G}$.

This data is assumed to satisfy the following axioms

- **Normalisation.** If $B \subseteq D$, and $\mathcal{F}$ is the unique element in $\mathsf{Mns}(B, B)$, then $F_\mathcal{F} = \mathrm{id}_{C_\mathcal{F}}$.
- **Transitivity.** For any $B' \subseteq B$ and $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{Mns}(B, B')$, $\Upsilon_{\mathcal{H}\mathcal{F}} = \Upsilon_{\mathcal{H}\mathcal{G}} \circ \Upsilon_{\mathcal{G}\mathcal{F}}$ as isomorphisms $F_\mathcal{F} \Rightarrow F_\mathcal{H}$. In particular, $\Upsilon_{\mathcal{F}\mathcal{F}} = \mathrm{id}_{F_\mathcal{F}}$ and $\Upsilon_{\mathcal{G}\mathcal{F}} = \Upsilon_{\mathcal{F}\mathcal{G}}^{-1}$.
- **Vertical factorisation.** For any $B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$ and $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, the tensor functor $F_{\mathcal{F}'\cup\mathcal{F}} : \mathcal{C}_B \to \mathcal{C}_{B''}$ is equal to the composition $F_{\mathcal{F}'} \circ F_\mathcal{F}$. Moreover, for any $\mathcal{G} \in \mathsf{Mns}(B, B')$ and $\mathcal{G}' \in \mathsf{Mns}(B', B'')$, the following equality holds
$$\Upsilon_{\mathcal{G}'\cup\mathcal{G}\,\mathcal{F}'\cup\mathcal{F}} = \begin{array}{c}\Upsilon_{\mathcal{G}\mathcal{F}}\\ \circ\\ \Upsilon_{\mathcal{G}'\mathcal{F}'}\end{array}$$
as isomorphisms $F_{\mathcal{F}'\cup\mathcal{F}} = F_{\mathcal{F}'} \circ F_\mathcal{F} \Rightarrow F_{\mathcal{G}'} \circ F_\mathcal{G} = F_{\mathcal{G}'\cup\mathcal{G}}$.[19]

14.2. **Morphisms.** Let $\mathcal{C}, \mathcal{C}'$ be two braided pre–Coxeter categories of type D. A 1–morphism $H : \mathcal{C} \to \mathcal{C}'$ consists of the following data.

- For any $B \subseteq D$, a braided tensor functor $H_B : \mathcal{C}_B \to \mathcal{C}'_B$.
- For any $B' \subseteq B \subseteq D$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, an isomorphism of tensor functors $\gamma_\mathcal{F} : F'_\mathcal{F} \circ H_B \Rightarrow H_{B'} \circ F_\mathcal{F}$ such that $\Upsilon_{\mathcal{G}\mathcal{F}} \circ \gamma_\mathcal{F} = \gamma_\mathcal{G} \circ \Upsilon'_{\mathcal{G}\mathcal{F}}$ as isomorphisms $F'_\mathcal{F} \circ H_B \Rightarrow H_{B'} \circ F_\mathcal{G}$.

This data is assumed to satisfy the following axioms.

- **Normalisation.** If $B \subseteq D$ and $\mathcal{F}$ is the unique element in $\mathsf{Mns}(B, B)$, so that $F_\mathcal{F} = \mathrm{id}_{\mathcal{C}_B}$ and $F'_\mathcal{F} = \mathrm{id}_{\mathcal{C}'_B}$, then $\gamma_\mathcal{F} = \mathrm{id}_{H_B}$.
- **Vertical factorisation.** For any $B'' \subseteq B' \subseteq B$, $\mathcal{F} \in \mathsf{Mns}(B, B')$ and $\mathcal{F}' \in \mathsf{Mns}(B', B'')$, the following equality holds
$$\gamma_{\mathcal{F}'\cup\mathcal{F}} = \begin{array}{c}\gamma_\mathcal{F}\\ \circ\\ \gamma_{\mathcal{F}'}\end{array}$$
as isomorphisms $F'_{\mathcal{F}'\cup\mathcal{F}} \circ H_B = F'_{\mathcal{F}'} \circ F'_\mathcal{F} \circ H_B \Rightarrow H_{B''} \circ F_{\mathcal{F}'} \circ F_\mathcal{F} = H_{B''} \circ F_{\mathcal{F}'\cup\mathcal{F}}$.

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[19] In [2], a more general version of vertical factorisation is considered, where the equalities $F_{\mathcal{F}'\cup\mathcal{F}} = F_{\mathcal{F}'} \circ F_\mathcal{F}$ and $\Upsilon_{\mathcal{G}'\cup\mathcal{G}\,\mathcal{F}'\cup\mathcal{F}} = \begin{array}{c}\Upsilon_{\mathcal{G}\mathcal{F}}\\ \Upsilon_{\mathcal{G}'\mathcal{F}'}\end{array}$ are only assumed to hold up to coherent isomorphisms. For the purposes of the present paper, it is sufficient to assume that these isomorphisms are equalities.

Let H^1, H^2 be two 1–morphisms $\mathcal{C} \to \mathcal{C}'$. A 2–morphism $v : H^1 \Rightarrow H^2$ consists of the following data.

- For any $B \subseteq D$, a natural transformation of braided tensor functors $v_B : H_B^1 \Rightarrow H_B^2$ such that, for any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, $\gamma_{\mathcal{F}} \circ v_B = v_{B'} \circ \gamma_{\mathcal{F}}$ as morphisms $F'_{\mathcal{F}} \circ H_B^1 \Rightarrow H_{B'}^2 \circ F_{\mathcal{F}}$.

14.3. **Generalised braid groups** [5]**.**

Definition. A *labeling* $\underline{m}$ of the diagram D is the assignment of an integer $m_{ij} \in \{2, 3, \ldots, \infty\}$ to any pair i, j of distinct vertices of D such that

$$m_{ij} = m_{ji} \qquad \text{and} \qquad m_{ij} = 2 \ \text{ if } i \text{ and } j \text{ are orthogonal}$$

The *generalised braid group (or Artin group)* corresponding to D and a labeling $\underline{m}$ is the group $B_D^{\underline{m}}$ generated by S_i, $i \in D$, with relations[20]

$$\underbrace{\mathsf{S}_i \cdot \mathsf{S}_j \cdot \mathsf{S}_i \ \cdots}_{m_{ij}} = \underbrace{\mathsf{S}_j \cdot \mathsf{S}_i \cdot \mathsf{S}_j \ \cdots}_{m_{ij}} \tag{14.1}$$

14.4. **Braided Coxeter categories.** Let $\underline{m}$ be a labeling of D. A *braided Coxeter category* of type $(D, \underline{m})$ is a braided pre–Coxeter category $(\mathcal{C}_B, F_{\mathcal{F}}, \Upsilon_{\mathcal{G}\mathcal{F}})$ of type D endowed with distinguished isomorphisms $S_i^{\mathcal{C}} \in \mathrm{Aut}(F_{\emptyset i})$ for any vertex i of D called *local monodromies*. These are assumed to satisfy the following.

- **Braid relations.** For any $B \subseteq D$, $i \neq j \in B$ and maximal nested sets $\mathcal{F}, \mathcal{G}$ on B with $\{i\} \in \mathcal{F}, \{j\} \in \mathcal{G}$, the following holds in $\mathsf{Aut}(F_{\mathcal{G}})$

$$\underbrace{\mathsf{Ad}\left(\Upsilon_{\mathcal{G}\mathcal{F}}\right)(S_i^{\mathcal{C}}) \cdot S_j^{\mathcal{C}} \cdot \mathsf{Ad}\left(\Upsilon_{\mathcal{G}\mathcal{F}}\right)(S_i^{\mathcal{C}}) \cdots}_{m_{ij}} = \underbrace{S_j^{\mathcal{C}} \cdot \mathsf{Ad}\left(\Upsilon_{\mathcal{G}\mathcal{F}}\right)(S_i^{\mathcal{C}}) \cdot S_j^{\mathcal{C}} \cdots}_{m_{ij}}$$

- **Coproduct identity.** For any $i \in D$, the following holds in $\mathrm{Aut}(F_{\emptyset i} \otimes F_{\emptyset i})$

$$J_i^{-1} \circ F_{\emptyset i}(c_i) \circ \Delta(S_i^{\mathcal{C}}) \circ J_i = c_{\emptyset} \circ S_i^{\mathcal{C}} \otimes S_i^{\mathcal{C}} \tag{14.2}$$

  where J_i is the tensor structure on $F_{\emptyset i}$ and $c_i, c_{\emptyset}$ are the opposite braidings in $\mathcal{C}_i$ and $\mathcal{C}_{\emptyset}$, respectively.[21] In other words, the following diagram is commutative for any $V, W \in \mathcal{C}_i$,

$$\begin{tikzcd}
F_{\emptyset i}(V) \otimes F_{\emptyset i}(W) \arrow[r, "S_i^V \otimes S_i^W"] \arrow[d, "J_i^{V,W}"'] & F_{\emptyset i}(V) \otimes F_{\emptyset i}(W) \arrow[r, "c_{\emptyset}"] & F_{\emptyset i}(W) \otimes F_{\emptyset i}(V) \arrow[d, "J_i^{W,V}"] \\
F_{\emptyset i}(V \otimes W) \arrow[r, "S_i^{V \otimes W}"'] & F_{\emptyset i}(V \otimes W) \arrow[r, "F_{\emptyset i}(c_i)"'] & F_{\emptyset i}(W \otimes V)
\end{tikzcd}$$

A 1–morphism $\mathcal{C} \to \mathcal{C}'$ of braided Coxeter categories is a 1–morphism of the underlying braided pre–Coxeter categories which preserves the local monodromies. That is, it consists of the data $(H_B, \gamma_{B'B})$ defined in 14.2 and such that, for any $i \in D$, $S_i^{\mathcal{C}} \circ \gamma_{\emptyset i} = \gamma_{\emptyset i} \circ S_i^{\mathcal{C}'}$ as isomorphisms $F'_i \circ H_i \Rightarrow H_{\emptyset} \circ F_i$.

A 2–morphism $H^1 \Rightarrow H^2$ of 1–morphisms $H^1, H^2 : \mathcal{C} \to \mathcal{C}'$ of braided Coxeter categories is a 2–morphism of the 1–morphisms of braided pre–Coxeter categories.

[20]The group $B_D^{\underline{m}}$ is called an Artin group in [6]. We follow here the terminology of [8].

[21] In a braided monoidal category with braiding β, the opposite braiding is $\beta^{\mathrm{op}}_{X,Y} := \beta^{-1}_{Y,X}$.

14.5. **Braid group representations.** The axioms of a braided Coxeter category are tailored to produce natural representations of the generalised braid group. More precisely, in [2, Prop. 3.9] we show that a braided Coxeter category $\mathcal{C}$ of type $(D, \underline{m})$ gives rise to a family of actions $\lambda_{\mathcal{F}} : B_D^{\underline{m}} \to \mathsf{Aut}(F_{\mathcal{F}})$ on the functors $F_{\mathcal{F}} : \mathcal{C}_D \to \mathcal{C}_\emptyset$ labeled by maximal nested sets on D, which are uniquely determined by the conditions

(1) $\lambda_{\mathcal{F}}(\mathsf{S}_i) = S_i^{\mathcal{C}}$ if $\{i\} \in \mathcal{F}$,
(2) $\lambda_{\mathcal{G}} = \mathsf{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}}) \circ \lambda_{\mathcal{F}}$.

14.6. **Deformation Drinfeld–Yetter modules.** We retain the notations from Section 12. Let $\mathbb{S}$ be a diagrammatic semigroup with underlying diagram D and $\mathfrak{b}$ an $\mathbb{S}$–graded Lie bialgebra. Let $\mathsf{DY}_{\mathfrak{b}}^{\hbar}$ be the category of Drinfeld–Yetter $\mathfrak{b}$–modules in the category of topologically free $\mathsf{k}[\![\hbar]\!]$–modules, $\widehat{\mathcal{U}}_{\mathfrak{b}}^{n}$ the algebra of endomorphisms of the n–fold forgetful functor $\mathsf{f} : \mathsf{DY}_{\mathfrak{b}}^{\hbar} \to \mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$, and $\mathfrak{U}_{\mathbb{S}}^{n}$ the universal algebra introduced in 12.11. Following the same procedure described in 7.4, one can rely on the category $\mathsf{DY}_{\mathfrak{b}_\hbar}^{\mathrm{adm}}$ of Drinfeld–Yetter modules over the Lie bialgebra $\mathfrak{b}_\hbar = (\mathfrak{b}[\![\hbar]\!], [\cdot,\cdot], \hbar\delta)$ whose coaction is divisible by $\hbar$ to obtain a homomorphism $\widehat{\rho}_{\mathfrak{b}}^{n} : \mathfrak{U}_{\mathbb{S}}^{n} \to \widehat{\mathcal{U}}_{\mathfrak{b}}^{n}$ which naturally extends to $\widehat{\mathfrak{U}}_{\mathbb{S}}^{n}$.

14.7. **From universal algebras to Drinfeld–Yetter modules.**

Proposition. *Let $\mathfrak{b}$ be an $\mathbb{S}$–graded Lie bialgebra.*

(1) *A braided pre–Coxeter structure $\mathcal{C}$ on $\widehat{\mathfrak{U}}_{\mathbb{S}}^{\bullet}$ canonically induces a braided pre–Coxeter structure $\mathcal{C}(\mathfrak{b})$ on $\{\mathsf{DY}_{\mathfrak{b}_B}^{\hbar}\}_{B\subseteq D}$.*
(2) *A twist $\mathcal{T}$ in $\widehat{\mathfrak{U}}_{\mathbb{S}}^{\bullet}$ canonically induces a 1–isomorphism $\mathcal{T}(\mathfrak{b}) : \mathcal{C}(\mathfrak{b}) \to \mathcal{C}_{\mathcal{T}}(\mathfrak{b})$, where $\mathcal{C}_{\mathcal{T}}$ denotes the twisted braided pre–Coxeter structure.*
(3) *A gauge g in $\widehat{\mathfrak{U}}_{\mathbb{S}}^{\bullet}$ canonically induces a 2–isomorphism $g(\mathfrak{b}) : \mathcal{T}(\mathfrak{b}) \Rightarrow \mathcal{T}_g(\mathfrak{b})$, where $\mathcal{T}_g$ denotes the gauged twist.*

Proof. (1) Let $\mathcal{C} = (\Phi_B, J_{\mathcal{F}}, \Upsilon_{\mathcal{F}\mathcal{G}})$ be a braided pre–Coxeter structure on $\widehat{\mathfrak{U}}_{\mathbb{S}}^{\bullet}$. We show below that the homomorphisms $\widehat{\rho}_{\mathfrak{b}}^{n}$ define a braided pre–Coxeter structure $\mathcal{C}(\mathfrak{b})$ with underlying categories $\{\mathsf{DY}_{\mathfrak{b}_B}^{\hbar}\}_{B\subseteq D}$.

- *Diagrammatic categories.* For any $B \subseteq D$, set $\mathcal{C}(\mathfrak{b})_B = \mathsf{DY}_{\mathfrak{b}_B}^{\Phi_B}$, the braided monoidal category of topologically free Drinfeld–Yetter $\mathfrak{b}_B$–modules, with associativity and commutativity constraints given by $\widehat{\rho}_B^{3}(\Phi_B)$ and $\widehat{\rho}_B^{2}(R_B)$ respectively.
- *Restriction functors.* For any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, let $F_{\mathcal{F}}^{\mathcal{C}}$ be the standard restriction functor $\mathrm{Res}_{B'B} = \mathrm{Res}_{\mathfrak{b}_{B'},\mathfrak{b}_B}$ with tensor structure $\widehat{\rho}_{BB'}^{2}(J_{\mathcal{F}})$.
- *De Concini–Procesi associators.* For any $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, let $\Upsilon_{\mathcal{G}\mathcal{F}}^{\mathcal{C}} : F_{\mathcal{F}}^{\mathcal{C}} \Rightarrow F_{\mathcal{G}}^{\mathcal{C}}$ be the tensor isomorphism defined by $\widehat{\rho}_{BB'}(\Upsilon_{\mathcal{G}\mathcal{F}})$.

We now show that this datum satisfies the properties required in 14.1.

- *Normalisation.* If $B \subseteq D$, and $\mathcal{F}$ is the unique element in $\mathsf{Mns}(B, B)$, then, by the vertical factorisation property of the relative twists in $\widehat{\mathfrak{U}}_{\mathbb{S}}^{\bullet}$ (cf. Definition 13.1), $J_{\mathcal{F}} = J_{\mathcal{F}} \cdot J_{\mathcal{F}}$. In particular, $J_{\mathcal{F}} = 1$ and $F_{\mathcal{F}} = \mathrm{id}_{\mathcal{C}(\mathfrak{b})_B}$.
- *Transitivity.* This follows from the horizontal factorisation of De Concini–Procesi associators in $\widehat{\mathfrak{U}}_{\mathbb{S}}^{\bullet}$.

- *Vertical factorisation.* This follows from the vertical factorisation of the relative twists and De Concini–Procesi associators in $\widehat{\mathfrak{U}}^\bullet_{\mathbb{S}}$.

(2) Let $\mathcal{T}=(u,F)$ be a twist in $\widehat{\mathfrak{U}}^\bullet_{\mathbb{S}}$ and $\mathcal{C}'=\mathcal{C}_{(u,F)}$ the twisted pre–Coxeter structure (cf. 13.2). Define a 1–isomorphism $\mathcal{T}(\mathfrak{b})=(H^{\mathcal{T}}_B,\gamma^{\mathcal{T}}_{\mathcal{F}}):\mathcal{C}(\mathfrak{b})\to\mathcal{C}'(\mathfrak{b})$ as follows.

- For any $B\subseteq D$, we denote by H_B the identity functor on $\mathcal{C}(\mathfrak{b})_B$ endowed with the tensor structure $\widehat{\rho}^2_B(F_B)$. In particular, it follows immediately from Definition 13.2 that H_B is a braided tensor equivalence $\mathcal{C}(\mathfrak{b})_B\to\mathcal{C}'(\mathfrak{b})_B$.
- For any $B'\subseteq B\subseteq D$ and $\mathcal{F}\in\mathsf{Mns}(B,B')$, we denote by $\gamma^{\mathcal{T}}_{\mathcal{F}}$ the natural isomorphism $F^{\mathcal{C}'}_{\mathcal{F}}\circ H^{\mathcal{T}}_B\Rightarrow H^{\mathcal{T}}_{B'}\circ F^{\mathcal{C}}_{\mathcal{F}}\circ H$ induced by $\widehat{\rho}_{BB'}(u_{\mathcal{F}})$. Therefore, by definition of u, $\gamma^{\mathcal{T}}_{\mathcal{F}}$ is a well–defined isomorphism of tensor functors satisfying the vertical factorisation property.

(3) Finally, let g be a gauge in $\widehat{\mathfrak{U}}^\bullet_{\mathbb{S}}$ and $\mathcal{T}'=\mathcal{T}_g$ the gauged twist (cf. 13.3). Then, we define a 2–isomorphism $g(\mathfrak{b}):\mathcal{T}(\mathfrak{b})\Rightarrow\mathcal{T}'(\mathfrak{b})$ as follows. For any $B\subseteq D$, we denote by v^g_B the isomorphism of braided tensor functors $H^{\mathcal{T}}_B\Rightarrow H^{\mathcal{T}'}_B$ given by $\widehat{\rho}_B(g_B)$. Then, it follows from the definition of g that $\gamma^{\mathcal{T}'}_{\mathcal{F}}\circ v^g_B=v^g_{B'}\circ\gamma^{\mathcal{T}}_{\mathcal{F}}$. □

14.8. **Universal braided pre–Coxeter structures.**

Definition. Let $\mathfrak{b}$ be an $\mathbb{S}$–graded Lie bialgebra. A braided pre–Coxeter structure (resp. 1–morphism, 2–morphism) on $\{\mathsf{DY}^{\hbar}_{\mathfrak{b}_B}\}_{B\subseteq D}$ is *universal* if it is induced by a braided pre–Coxeter structure (resp. twist, gauge) on $\widehat{\mathfrak{U}}^\bullet_{\mathbb{S}}$ via Proposition 14.7.

The following is a direct consequence of Theorem 13.4.

Theorem. *Let $\mathfrak{b}$ be an $\mathbb{S}$–graded Lie bialgebra, and $\mathcal{C}_1,\mathcal{C}_2$ two universal braided pre–Coxeter structures with diagrammatic categories $\{\mathsf{DY}^{\hbar}_{\mathfrak{b}_B}\}_{B\subseteq D}$. Then, there is a universal 1–isomorphism $\mathcal{C}_1\to\mathcal{C}_2$, which is unique up to a unique universal 2–isomorphism.*

In [2, Thm. 9.1], we show that, for any $\mathbb{S}$–graded Lie bialgebra $\mathfrak{b}$, there is a canonical universal braided pre–Coxeter structure on Drinfeld–Yetter modules. Although we do not need this result for the purposes of this paper, we observe that, combined with the uniqueness result above, this implies the following.

Corollary. *Let $\mathfrak{b}$ be an $\mathbb{S}$–graded Lie bialgebra. Then, there exists an essentially unique universal braided pre–Coxeter structure on the categories of deformation Drinfeld–Yetter modules.*

15. Coxeter structures and Kac–Moody algebras

In this section, we consider the diagrammatic semigroup of positive roots of a symmetrisable Kac–Moody algebra $\mathfrak{g}$ with negative Borel subalgebra $\mathfrak{b}$. We then rely on the results of Sections 13–14 to prove the uniqueness of braided Coxeter structures on integrable Drinfeld–Yetter modules over $\mathfrak{b}$, and category $\mathcal{O}$ modules over $\mathfrak{g}$.

15.1. **Kac–Moody algebras** [19]**.** Throughout this section, we fix a finite set $\mathbf{I}$, a matrix $\mathsf{A} = (a_{ij})_{i,j\in\mathbf{I}}$ with entries in k, and a realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A. Thus, $\mathfrak{h}$ is a k–vector space of dimension $2|\mathbf{I}| - \mathrm{rk}(\mathsf{A})$, and $\Pi = \{\alpha_i\}_{i\in\mathbf{I}} \subset \mathfrak{h}^*$, $\Pi^\vee = \{\alpha_i^\vee\}_{i\in\mathbf{I}} \subset \mathfrak{h}$ are linearly independent subsets such that $\alpha_i(\alpha_j^\vee) = a_{ji}$.

Let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}(\mathsf{A})$ be the Lie algebra generated by $\mathfrak{h}$, $\{e_i, f_i\}_{i\in\mathbf{I}}$ with relations $[h, h'] = 0$, for any $h, h' \in \mathfrak{h}$, and

$$[h, e_i] = \alpha_i(h)e_i \qquad [h, f_i] = -\alpha_i(h)f_i \qquad [e_i, f_j] = \delta_{ij}\alpha_i^\vee$$

The Kac–Moody algebra corresponding to A is the Lie algebra $\mathfrak{g} = \mathfrak{g}(\mathsf{A}) = \widetilde{\mathfrak{g}}/I$, where I is the sum of all two–sided ideals in $\widetilde{\mathfrak{g}}$ having trivial intersection with $\mathfrak{h} \subset \widetilde{\mathfrak{g}}$. If A is a generalised Cartan matrix (*i.e.,* $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\leqslant 0}$ if $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$), the ideal I is generated by the Serre relations $\mathsf{ad}(e_i)^{1-a_{ij}}(e_j) = 0 = \mathsf{ad}(f_i)^{1-a_{ij}}(f_j)$ for any $i \neq j$.

Set $\mathsf{Q}_+ = \bigoplus_{i\in\mathbf{I}} \mathbb{Z}_{\geqslant 0}\alpha_i \subseteq \mathfrak{h}^*$, so that $\mathfrak{g}$ has the root space decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}+$, where $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\mathsf{Q}_+\setminus\{0\}} \mathfrak{g}_{\pm\alpha}$, and $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [h, X] = \alpha(h)X,\ \forall h \in \mathfrak{h}\}$. Denote by $\mathsf{R}_+ = \{\alpha \in \mathsf{Q}_+ \mid \mathfrak{g}_\alpha \neq 0\}$ the set of positive roots of $\mathfrak{g}$.

15.2. **Extended Kac–Moody algebras** [2]**.** Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}(\mathsf{A})$ be the Lie algebra generated by $\{e_i, f_i, \alpha_i^\vee, \lambda_i^\vee\}_{i\in\mathbf{I}}$ with relations $[\alpha_i^\vee, \alpha_j^\vee] = [\lambda_i^\vee, \lambda_j^\vee] = [\alpha_i^\vee, \lambda_j^\vee] = 0$ for any $i, j \in \mathbf{I}$, and

$$[\alpha_i^\vee, e_j] = a_{ij}e_j \qquad [\alpha_i^\vee, f_j] = -a_{ij}f_j \qquad [\lambda_i^\vee, e_j] = \delta_{ij}e_j \qquad [\lambda_i^\vee, f_j] = -\delta_{ij}f_j$$

Definition. The *extended Kac–Moody algebra* corresponding to A is the k–Lie algebra $\overline{\mathfrak{g}} = \widehat{\mathfrak{g}}/I$, where I is the sum of all two–sided ideals in $\widehat{\mathfrak{g}}$ having trivial intersection with the abelian subalgebra $\overline{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$ spanned by $\{\alpha_i^\vee, \lambda_i^\vee\}_{i\in\mathbf{I}}$.

Let D be the Dynkin diagram of $\mathfrak{g}$ and, for any $B \subseteq D$, let $\overline{\mathfrak{g}}_B \subseteq \overline{\mathfrak{g}}$ be the Lie subalgebra generated by $\{e_i, f_i, \alpha_i^\vee, \lambda_i^\vee\}_{i\in B}$ if $B \neq \emptyset$, and $\overline{\mathfrak{g}}_\emptyset = \{0\}$ otherwise.

Proposition. *The extended Kac–Moody algebra $\overline{\mathfrak{g}}$ is a diagrammatic Lie algebra with Lie subalgebras $\overline{\mathfrak{g}}_B$, $B \subseteq D$.*

PROOF. Clearly, for any $B_1 \subseteq B_2$, $\overline{\mathfrak{g}}_{B_1} \subseteq \overline{\mathfrak{g}}_{B_2}$. If $B_3 \perp B_4$, then for any $i \in B_3, j \in B_4$, e_i, f_i commute with e_j, f_j [19, Lemma 1.6], and, since $[\alpha_i^\vee, e_j] = 0 = [\alpha_i^\vee, f_j]$ and $[\lambda_i^\vee, e_j] = 0 = [\lambda_i^\vee, f_j]$, $[\overline{\mathfrak{g}}_{B_3}, \overline{\mathfrak{g}}_{B_4}] = 0$. Finally, if $B = B_1 \sqcup B_2$, $\overline{\mathfrak{g}}_B = \overline{\mathfrak{g}}_{B_1} \oplus \overline{\mathfrak{g}}_{B_2}$. □

Remark. The definition of $\overline{\mathfrak{g}}$ takes its cue from [17]. Its use is prompted by the fact that not all (symmetrisable) Kac–Moody algebras are diagrammatic [2, 11].

15.3. **Relation between $\mathfrak{g}$ and $\overline{\mathfrak{g}}$.** We show in [2, §11.6] that $\overline{\mathfrak{g}}$ is non–canonically a split central extension of $\mathfrak{g}$, with a $\mathrm{rk}(\mathsf{A})$–dimensional kernel. Namely, set $r = \mathrm{rk}\,\mathsf{A}$, $\ell = |\mathbf{I}|$, and assume for simplicity that the first r rows of A are linearly independent. Let $\mathfrak{h}' \subset \mathfrak{h}$ be the (ℓ–dimensional) span of $\{\alpha_i^\vee\}_{i\in\mathbf{I}}$, and $\mathfrak{h}'' \subset \mathfrak{h}$ a subspace with basis $\{d_j\}_{j=r+1}^{\ell}$ such that $\alpha_i(d_j) = \delta_{ij}$, $1 \leqslant i \leq \ell$, $r+1 \leqslant j \leq \ell$. Let $\varpi_i^\vee = \sum_{j=1}^{r} c_{ij}\alpha_j^\vee$ be the fundamental coweights corresponding to $\{\alpha_1, \ldots, \alpha_r\}$. Then, the elements $\gamma_i \coloneqq \varpi_i^\vee - \lambda_i^\vee \in \overline{\mathfrak{h}}$, $i = 1, \ldots, r$, are central in $\overline{\mathfrak{g}}$. Denote by $\mathfrak{c}$ the subspace spanned by $\{\gamma_i\}_{i=1}^{r}$.

Proposition ([2])**.** *The choice of the complementary subspace $\mathfrak{h}'' \subset \mathfrak{h}$ determines*

(1) *An embedding $\mathfrak{g} \subset \overline{\mathfrak{g}}$, mapping $e_i, f_i, \alpha_i^\vee \mapsto e_i, f_i, \alpha_i^\vee$, and $d_j \mapsto \lambda_j^\vee$ for any $i \in \mathbf{I}, j = r+1, \ldots, \ell$.*

(2) *A projection* $\overline{\mathfrak{g}} \to \overline{\mathfrak{g}}/\mathfrak{c} \to \mathfrak{g}$ *mapping* $e_i, f_i, \alpha_i^\vee \to e_i, f_i, \alpha_i^\vee$, $i \in \mathbf{I}$, $\lambda_j^\vee \mapsto \varpi_j^\vee$, *if* $j = 1, \dots, r$, *and* $\lambda_j^\vee \mapsto d_j$, *if* $j = r+1, \dots, \ell$.
(3) *A Lie algebra isomorphism* $\overline{\mathfrak{g}} \simeq \mathfrak{g} \oplus \mathfrak{c}$.

15.4. **Root space decomposition of $\overline{\mathfrak{g}}$.** Let $\{\overline{\alpha}_i\}_{i\in\mathbf{I}}$ be the linear forms on $\overline{\mathfrak{h}}$ defined by

$$\overline{\alpha}_i(\alpha_j^\vee) = a_{ji} \qquad \text{and} \qquad \overline{\alpha}_i(\lambda_j^\vee) = \delta_{ij}$$

so that, for any $h \in \overline{\mathfrak{h}}$, $[h, e_i] = \overline{\alpha}_i(h)e_i$ and $[h, f_i] = -\overline{\alpha}_i(h)f_i$. Set $\overline{\mathsf{Q}}_+ = \bigoplus_{i\in\mathbf{I}} \mathbb{Z}_{\geqslant 0}\overline{\alpha}_i \subseteq \overline{\mathfrak{h}}^*$. Then, $\overline{\mathfrak{g}}$ has the root space decomposition

$$\overline{\mathfrak{g}} = \bigoplus_{\substack{\alpha\in\overline{\mathsf{Q}}_+\\ \alpha\neq 0}} \overline{\mathfrak{g}}_\alpha \oplus \overline{\mathfrak{h}} \oplus \bigoplus_{\substack{\alpha\in\overline{\mathsf{Q}}_+\\ \alpha\neq 0}} \overline{\mathfrak{g}}_{-\alpha}$$

where $\overline{\mathfrak{g}}_\alpha = \{X \in \overline{\mathfrak{g}} \mid [h, X] = \alpha(h)X \;\forall\, h \in \overline{\mathfrak{h}}\}$. Let $\bar{\cdot} : \mathsf{Q}_+ \to \overline{\mathsf{Q}}_+$ be the $\mathbb{Z}_{\geqslant 0}$–linear map sending α_i to $\overline{\alpha}_i$, $i \in \mathbf{I}$. It follows from the proposition above that, for any $\alpha \in \mathsf{Q}_+$, $\alpha \neq 0$, $\overline{\mathfrak{g}}_{\overline{\alpha}}$ identifies canonically with $\mathfrak{g}_\alpha$, and

$$\overline{\mathfrak{g}} = \bigoplus_{\alpha\in\mathsf{R}_+} \mathfrak{g}_\alpha \oplus \overline{\mathfrak{h}} \oplus \bigoplus_{\alpha\in\mathsf{R}_+} \mathfrak{g}_{-\alpha}$$

15.5. **Symmetrisable Kac–Moody algebras.** Assume henceforth that the matrix A is symmetrisable, and fix an invertible diagonal $\mathsf{D} = \mathrm{Diag}(d_i)_{i\in\mathbf{I}}$ such that AD is symmetric.

Let $\mathfrak{h}' \subset \mathfrak{h}$ be the span of $\{\alpha_i^\vee\}_{i\in\mathbf{I}}$, and $\mathfrak{h}'' \subset \mathfrak{h}$ a complementary subspace. Then, there is a symmetric, non–degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$, which is uniquely determined by $\langle \alpha_i^\vee, \cdot\rangle = d_i\alpha_i(\cdot)$ and $\langle \mathfrak{h}'', \mathfrak{h}''\rangle = 0$. The form $\langle\cdot,\cdot\rangle$ uniquely extends to an invariant symmetric bilinear form on $\widetilde{\mathfrak{g}}$, and $\langle e_i, f_j\rangle = \delta_{ij}d_i$. The kernel of this form is precisely I, so that $\langle\cdot,\cdot\rangle$ descends to a nondegenerate form on $\mathfrak{g}$.

Set $\mathfrak{b}_\pm = \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathsf{R}_+} \mathfrak{g}_{\pm\alpha} \subset \mathfrak{g}$. The bilinear form induces a canonical isomorphism of graded vector spaces $\mathfrak{b}_+ \simeq \mathfrak{b}_-^\star$, where $\mathfrak{b}_-^\star = \mathfrak{h}^* \oplus \bigoplus_{\alpha\in\mathsf{R}_+} \mathfrak{g}_{-\alpha}^*$, and determines on $\mathfrak{g}$ a natural structure of Lie bialgebra with cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ given by

$$\delta|_{\mathfrak{h}} = 0 \qquad \delta(e_i) = d_i^{-1}\alpha_i^\vee \wedge e_i \qquad \delta(f_i) = d_i^{-1}\alpha_i^\vee \wedge f_i$$

15.6. **Extended symmetrisable Kac–Moody algebras.** The extended Lie algebra $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(\mathsf{A})$ is endowed with an invariant, symmetric and non–degenerate bilinear form $\langle\cdot,\cdot\rangle : \overline{\mathfrak{g}} \otimes \overline{\mathfrak{g}} \to \mathsf{k}$ uniquely determined by the table

$\langle\cdot,\cdot\rangle$	e_j	$\alpha_j^\vee$	$\lambda_j^\vee$	f_j
e_i	0	0	0	$\delta_{ij}d_i$
$\alpha_i^\vee$	0	d_ja_{ij}	$\delta_{ij}d_i$	0
$\lambda_i^\vee$	0	$\delta_{ij}d_j$	0	0
f_i	$\delta_{ij}d_j$	0	0	0

If the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ given in 15.5 is obtained from a subspace $\mathfrak{h}'' \subset \mathfrak{h}$ satisfying the requirements of 15.3, the embedding $\mathfrak{g} \subset \overline{\mathfrak{g}}$ corresponding to $\mathfrak{h}''$ is compatible with the bilinear forms and the Lie bialgebra structures.

There is a natural structure of Lie bialgebra on $\overline{\mathfrak{g}}$ with cobracket $\delta : \overline{\mathfrak{g}} \to \overline{\mathfrak{g}} \wedge \overline{\mathfrak{g}}$

$$\delta|_{\overline{\mathfrak{h}}} = 0 \qquad \delta(e_i) = d_i^{-1}\alpha_i^\vee \wedge e_i \qquad \delta(f_i) = d_i^{-1}\alpha_i^\vee \wedge f_i$$

15.7. **Drinfeld double realisation.** It is well–known that any symmetrisable Kac–Moody algebra is a central quotient of the restricted Drinfeld double of its Borel subalgebra. An analogous result holds for a symmetrisable extended Kac–Moody algebra $\overline{\mathfrak{g}}$ [2, 10.7]. Specifically, consider the Lie algebra $\overline{\mathfrak{g}}^{(2)} = \overline{\mathfrak{g}} \oplus \overline{\mathfrak{h}}^c$, where $\overline{\mathfrak{h}}^c = \overline{\mathfrak{h}}$ is central in $\overline{\mathfrak{g}}^{(2)}$, and endow it with the inner product $\langle \cdot, \cdot \rangle \oplus - \langle \cdot, \cdot \rangle|_{\overline{\mathfrak{h}}^c \times \overline{\mathfrak{h}}^c}$. Let $\pi_0 : \overline{\mathfrak{g}} \to \overline{\mathfrak{h}}$ be the projection, and $\overline{\mathfrak{b}}^{(2)}_{\pm} \subset \overline{\mathfrak{g}}^{(2)}$ the subalgebra

$$\overline{\mathfrak{b}}^{(2)}_{\pm} = \{(X, h) \in \overline{\mathfrak{b}}_{\pm} \oplus \overline{\mathfrak{h}}^c \,|\, \pi_0(X) = \pm h\}$$

The projection $\overline{\mathfrak{g}}^{(2)} \to \overline{\mathfrak{g}}$ onto the first component restricts to an isomorphism $\overline{\mathfrak{b}}^{(2)}_{\pm} \to \overline{\mathfrak{b}}_{\pm}$ with inverse $\overline{\mathfrak{b}}_{\pm} \ni X \to (X, \pm\pi_0(X)) \in \overline{\mathfrak{b}}^{(2)}_{\pm}$. Then, it is easy to see that $\overline{\mathfrak{g}}^{(2)} = \overline{\mathfrak{g}} \oplus \overline{\mathfrak{h}}^c$ is the restricted Drinfeld double of $\overline{\mathfrak{b}}^{(2)}_{-} \simeq \overline{\mathfrak{b}}_{-}$.

15.8. **Diagrammatic semigroup structure.** Let $\mathbb{S} = (\mathsf{S}, D)$ be the diagrammatic semigroup introduced in 12.8. Thus, D is the Dynkin diagram of $\mathfrak{g}$, $\mathsf{S} = \mathsf{R}_+$ its partial semigroup of positive roots, and $\mathsf{S}(B) = \mathsf{R}_{B,+}$ for any $B \subseteq D$.

For any $B \subseteq D$, let $\overline{\mathfrak{b}}_{B,-}$ (resp. $\overline{\mathfrak{b}}_{B,+}$) be the Lie subbialgebra of $\overline{\mathfrak{g}}_B$ generated by $\{\alpha_i^\vee, \lambda_i^\vee, f_i\}_{i \in B}$ (resp. $\{\alpha_i^\vee, \lambda_i^\vee, e_i\}_{i \in B}$). Then,

$$\overline{\mathfrak{b}}_{B,\pm} = \overline{\mathfrak{h}}_B \oplus \overline{\mathfrak{n}}_{B,\pm}$$

where $\overline{\mathfrak{h}}_B \subseteq \overline{\mathfrak{h}}$ is spanned by $\{\alpha_i^\vee, \lambda_i^\vee\}_{i \in B}$, and $\overline{\mathfrak{n}}_{B,\pm} = \bigoplus_{\alpha \in \mathsf{R}_{B,+}} \overline{\mathfrak{g}}_{\pm\alpha}$, with $\mathsf{R}_{B,+} = \mathsf{R}_+ \cap \bigoplus_{i \in B} \mathbb{Z}_{\geqslant 0} \alpha_i$. For any $B' \subseteq B$, set

$$\overline{\mathfrak{h}}_B = \overline{\mathfrak{h}}_{B'} \oplus \overline{\mathfrak{h}}_{B'}^{\perp} \qquad \text{and} \qquad \overline{\mathfrak{n}}_{B,\pm} = \overline{\mathfrak{n}}_{B',\pm} \oplus \overline{\mathfrak{n}}_{B',\pm}^{\perp} \tag{15.1}$$

where $\overline{\mathfrak{h}}_{B'}^{\perp} \subseteq \{t \in \overline{\mathfrak{h}}_B \,|\, \alpha_i(t) = 0,\, i \in B'\}$ is a chosen complement to $\overline{\mathfrak{h}}_{B'}$ in $\overline{\mathfrak{h}}_B$, and $\overline{\mathfrak{n}}_{B',\pm}^{\perp} = \bigoplus_{\alpha \in \mathsf{R}_{B,+} \setminus \mathsf{R}_{B',+}} \overline{\mathfrak{g}}_{\pm\alpha}$. For any $B' \subseteq B$, let

$$i_{0,BB',\pm} : \overline{\mathfrak{h}}_{B'} \to \overline{\mathfrak{b}}_{B,\pm} \qquad\qquad p_{0,B'B,\pm} : \overline{\mathfrak{b}}_{B,\pm} \to \overline{\mathfrak{h}}_{B'}$$

and

$$i_{BB',\pm} : \overline{\mathfrak{b}}_{B',\pm} \to \overline{\mathfrak{b}}_{B,\pm} \qquad\qquad p_{B'B,\pm} : \overline{\mathfrak{b}}_{B,\pm} \to \overline{\mathfrak{b}}_{B',\pm}$$

be the canonical injections and the projections corresponding to the splitting (15.1). Set $\theta_{B,\pm} = i_{DB,\pm} \circ p_{BD,\pm}$ and $\theta_{0,B,\pm} = i_{0,DB,\pm} \circ p_{0,BD,\pm}$ in $\mathrm{End}(\overline{\mathfrak{b}}_{\pm})$. Then,

$$\theta_{B,\pm} = \theta_{0,B,\pm} + \sum_{\alpha \in \mathsf{R}_B^+} \theta_{\pm\alpha}$$

where $\theta_{\pm\alpha}$ denotes the standard idempotent projecting over the root space $\overline{\mathfrak{g}}_{\pm\alpha}$. The following is straightforward.

Proposition. *The data $(\theta_{\pm\alpha}, \theta_{0,B,\pm})$ induce an $\mathbb{S}$–graded Lie bialgebra structure on $\overline{\mathfrak{b}}_{\pm}$ (cf. 12.7). In particular, the maps $\theta_{B,\pm}$ are morphisms of Lie bialgebras and $\overline{\mathfrak{b}}_{\pm}$ is split diagrammatic with Lie subbialgebras $\overline{\mathfrak{b}}_{B,\pm}$, $B \subseteq D$.*

By Proposition 14.7, a braided pre–Coxeter structure $\mathcal{C}$ on $\widehat{\mathfrak{U}}^{\bullet}_{\mathbb{S}}$ induces a universal braided pre–Coxeter structure $\mathcal{C}(\overline{\mathfrak{b}}_{\pm})$ on Drinfeld–Yetter modules. Theorem 14.8 then yields the following

Corollary. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two universal braided pre–Coxeter structures on $\{\mathsf{DY}^{\hbar}_{\mathfrak{b}_B}\}_{B \subseteq D}$. Then, there is a universal 1–isomorphism $\mathcal{C}_1 \to \mathcal{C}_2$, which is unique up to a unique universal 2–isomorphism.*

15.9. **The category $\mathcal{O}_{\overline{\mathfrak{g}}}$.** A $\overline{\mathfrak{g}}$–module V is in category $\mathcal{O}_{\overline{\mathfrak{g}}}$ if the following holds.

($\mathcal{O}1$) $V=\bigoplus_{\lambda\in\overline{\mathfrak{h}}^*}V_\lambda$, where $V_\lambda=\{v\in V\,|\,h\,v=\lambda(h)v,\ h\in\overline{\mathfrak{h}}\}$

($\mathcal{O}2$) $\dim V_\lambda<\infty$ for any $\lambda\in\mathsf{P}(V)=\{\lambda\in\overline{\mathfrak{h}}^*|\,V_\lambda\neq 0\}$

($\mathcal{O}3$) $\mathsf{P}(V)\subseteq D(\lambda_1)\cup\cdots\cup D(\lambda_m)$, for some $\lambda_1,\dots,\lambda_m\in\overline{\mathfrak{h}}^*$

where $D(\lambda)=\{\mu\in\overline{\mathfrak{h}}^*\mid\mu\leqslant\lambda\}$, with $\mu\leqslant\lambda$ iff $\lambda-\mu\in\mathsf{Q}_+$. The category $\mathcal{O}_{\overline{\mathfrak{g}}}$ has a natural symmetric tensor structure inherited from $\mathrm{Rep}\,\overline{\mathfrak{g}}$.

We observed in 15.7 that the restricted Drinfeld double of the negative Borel subalgebra $\overline{\mathfrak{b}}_-$ of $\overline{\mathfrak{g}}$ is isomorphic to the trivial central extension $\overline{\mathfrak{g}}^{(2)}=\overline{\mathfrak{g}}\oplus\overline{\mathfrak{h}}^c$ of $\mathfrak{g}$ by $\overline{\mathfrak{h}}^c=\overline{\mathfrak{h}}$. It follows by 2.2–2.3 that the category of Drinfeld–Yetter modules over $\overline{\mathfrak{b}}_-$ is equivalent to the category $\mathcal{E}_{\overline{\mathfrak{g}}^{(2)}}$ of $\overline{\mathfrak{g}}^{(2)}$–modules, where $\overline{\mathfrak{g}}^{(2)}=\overline{\mathfrak{g}}\oplus\overline{\mathfrak{h}}^c$, which carry a locally finite action of $\overline{\mathfrak{b}}_+^{(2)}\subset\overline{\mathfrak{g}}^{(2)}$. This implies the following.

Proposition.

1. *The category $\mathcal{O}_{\overline{\mathfrak{g}}}$ is isomorphic to the full tensor subcategory of $\mathcal{E}_{\overline{\mathfrak{g}}^{(2)}}$ consisting of those modules carrying a trivial action of $\overline{\mathfrak{h}}^c$ and satisfying, as a module over $\overline{\mathfrak{h}}\subset\overline{\mathfrak{g}}\subset\overline{\mathfrak{g}}^{(2)}$, the conditions ($\mathcal{O}1$)–($\mathcal{O}3$) above.*
2. *Under the equivalence $\mathcal{E}_{\overline{\mathfrak{g}}^{(2)}}\simeq\mathsf{DY}_{\overline{\mathfrak{b}}_-}$, $\mathcal{O}_{\overline{\mathfrak{g}}}$ is isomorphic to the full tensor subcategory of $\mathsf{DY}_{\overline{\mathfrak{b}}_-}$ consisting of those modules V such that the action and the coaction of $\overline{\mathfrak{h}}$ on V coincide under $\langle\cdot,\cdot\rangle_{\overline{\mathfrak{h}}}$, i.e.,*
$$\pi_V\circ i_0\otimes\mathrm{id}_V=\langle\cdot,\cdot\rangle_{\overline{\mathfrak{h}}}\otimes\mathrm{id}_V\circ\mathrm{id}_{\overline{\mathfrak{h}}}\otimes p_0\otimes\mathrm{id}_V\otimes\mathrm{id}\circ\mathrm{id}_{\overline{\mathfrak{h}}}\otimes\pi_V^* \tag{15.2}$$
and, as a module over $\overline{\mathfrak{h}}\subset\overline{\mathfrak{b}}_-$, V satisfies the conditions ($\mathcal{O}1$)–($\mathcal{O}3$) above.

15.10. **Pre–Coxeter structures and category $\mathcal{O}_{\overline{\mathfrak{g}}}$.** Condition ($\mathcal{O}2$) on the finite–dimensionality of weight spaces in 15.9 is not stable under restriction from $\overline{\mathfrak{g}}=\overline{\mathfrak{g}}_D$ to $\overline{\mathfrak{g}}_B$ if $B\subsetneq D$, which makes category $\mathcal{O}_{\overline{\mathfrak{g}}}$ unsuitable to the axiomatic of braided pre–Coxeter structures. We therefore omit it, and denote by $\mathcal{O}_{\infty,\overline{\mathfrak{g}}}$ the category of $\overline{\mathfrak{g}}$–modules satisfying conditions ($\mathcal{O}1$) and ($\mathcal{O}3$). Proposition 15.9 shows that $\mathcal{O}_{\infty,\overline{\mathfrak{g}}}$ is a full subcategory of $\mathsf{DY}_{\overline{\mathfrak{b}}_-}$. Moreover, any universal braided pre–Coxeter structure on $\{\mathsf{DY}^\hbar_{\overline{\mathfrak{b}}_{B,-}}\}_{B\subseteq D}$ restricts to one on $\{\mathcal{O}^\hbar_{\infty,\overline{\mathfrak{g}}_B}\}_{B\subseteq D}$.

15.11. **Braid group actions.** Assume now that A is a symmetrisable generalised Cartan matrix, let W be the corresponding Weyl group with set of simple reflections $\{s_i\}_{i\in\mathbf{I}}$, and set $\underline{m}=(m_{ij})$, where m_{ij} is the order of s_is_j in W.

Let $\mathcal{C}^{\mathsf{int}}_{\overline{\mathfrak{g}}}$ be the category of integrable $\overline{\mathfrak{g}}$–modules, *i.e.,* $\overline{\mathfrak{h}}$–semisimple modules endowed with a locally nilpotent action of the elements $\{e_i,f_i\}_{i\in\mathbf{I}}$. Let $\widehat{\mathcal{C}}^{\mathsf{int}}_{\overline{\mathfrak{g}}}$ be the algebra $\mathsf{End}\left(\mathcal{C}^{\mathsf{int}}_{\overline{\mathfrak{g}}}\to\mathrm{Vect}\right)$ and, for any $i\in D$, denote by $\widetilde{s}_i\in\widehat{\mathcal{C}}^{\mathsf{int}}_{\overline{\mathfrak{g}}}$ the triple exponential
$$\widetilde{s}_i=\exp(e_i)\cdot\exp(-f_i)\cdot\exp(e_i).$$
It is well–known (cf. [24]) that these satisfy the generalised braid relations (14.1).

Let $\mathsf{DY}^{\mathsf{int},0}_{\overline{\mathfrak{b}}_-}$ be the category of integrable Drinfeld–Yetter $\overline{\mathfrak{b}}_-$–modules in $\mathsf{DY}_{\overline{\mathfrak{b}}_-}$, *i.e.,* $\overline{\mathfrak{h}}$–diagonalisable, endowed with a locally nilpotent action of the elements $\{f_i\}_{i\in D}\subseteq\overline{\mathfrak{b}}_-$, and satisfying (15.2), so as to give rise to integrable modules over $\overline{\mathfrak{g}}$.

In particular, the triple exponential $\widetilde{s}_i$ acts on the objects in $\mathsf{DY}^{\mathsf{int},0}_{\overline{\mathfrak{b}}_-}$ and the subcategory of integrable modules in $\mathcal{O}_{\infty,\overline{\mathfrak{g}}}$, denoted $\mathcal{O}^{\mathsf{int}}_{\infty,\overline{\mathfrak{g}}}$, is isomorphic to a braided tensor subcategory of $\mathsf{DY}^{\mathsf{int},0}_{\overline{\mathfrak{b}}_-}$.

15.12. **Universal braided Coxeter structures on Kac–Moody algebras.** Set $\overline{\mathfrak{b}}=\overline{\mathfrak{b}}_-$. Let $\mathsf{DY}^{\hbar,\mathsf{int},0}_{\overline{\mathfrak{b}}}$ be the category of integrable deformation Drinfeld–Yetter $\overline{\mathfrak{b}}$–modules. As usual, we denote by $\widehat{\mathcal{U}}^n_B$ (resp. $\widehat{\mathcal{U}}^n_{B,0}$) the algebra of endomorphisms of the forgetful functor $\mathsf{f}^{\boxtimes n}_B:(\mathsf{DY}^{\hbar}_{\mathfrak{b}_B})^n\to\mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$ (resp. $\mathsf{f}^{\boxtimes n}_{B,0}:(\mathsf{DY}^{\hbar}_{\mathfrak{b}_B})^n\to\mathsf{DY}^{\hbar}_{\mathfrak{h}}$). For any $X\in\widehat{\mathcal{U}}^n_B$, we denote by $\mathsf{p}(X)$ the induced endomorphism of the forgetful functor $(\mathsf{DY}^{\hbar,\mathsf{int},0}_{\mathfrak{b}_B})^n\to\mathsf{Vect}_{\mathsf{k}[\![\hbar]\!]}$.

Definition. A braided Coxeter structure of type $(D,\underline{m})$ with diagrammatic categories $\{\mathsf{DY}^{\hbar,\mathsf{int},0}_{\mathfrak{b}_B}\}_{B\subseteq D}$ is *good* (resp. *universal*) if the underlying braided pre–Coxeter structure is induced by a weight–zero[22] (resp. universal) pre–Coxeter structure on $\mathsf{DY}^{\hbar}_{\overline{\mathfrak{b}}}$, and its local monodromies have the form

$$S_i=\widetilde{s}_i\cdot\mathsf{p}(\underline{S}_i) \tag{15.3}$$

where $\underline{S}_i\in\widehat{\mathcal{U}}^1_{\{i\},0}$, $\underline{S}_i=1 \mod \hbar$, and $\widetilde{s}_i=\exp(e_i)\exp(-f_i)\exp(e_i)$.

Remark. It follows from Proposition 10.3 (2) that any universal braided Coxeter structure on $\{\mathsf{DY}^{\hbar,\mathsf{int},0}_{\mathfrak{b}_B}\}_{B\subseteq D}$ is good. Moreover, it is important to observe that, since $\mathsf{DY}^{\hbar}_{\mathfrak{b}_i}\simeq\mathrm{Rep}\,U\overline{\mathfrak{g}}^{(2)}_i[\![\hbar]\!]$ with $\mathfrak{g}_i=\mathfrak{sl}_2^{\alpha_i}$, we have $\widehat{\mathcal{U}}^n_{\{i\}}=(U\overline{\mathfrak{g}}^{(2)}_i)^{\otimes n}[\![\hbar]\!]$. In particular, $\mathsf{p}(\underline{S}_i)$ is an element in $(U\overline{\mathfrak{g}}_i)^{\overline{\mathfrak{h}}_i}[\![\hbar]\!]$.

The twisting of a braided pre–Coxeter structure on $\{\mathsf{DY}^{\hbar}_{\mathfrak{b}_B}\}_{B\subseteq D}$ extends to a twisting of a braided Coxeter structure on $\{\mathsf{DY}^{\hbar,\mathsf{int},0}_{\mathfrak{b}_B}\}_{B\subseteq D}$. Namely, if $(\mathcal{C},S_i)$ is a good braided Coxeter structure on the latter, where $\mathcal{C}$ is the corresponding braided pre–Coxeter structure on the former, and (u,F) is a weight–zero twist of $\mathcal{C}$, then

$$(\mathcal{C},S_i)_{(u,F)}\coloneqq(\mathcal{C}_{(u,F)},S^u_i\coloneqq u_{\{i\}}\cdot S_i\cdot u^{-1}_{\{i\}})$$

is a good braided Coxeter structure on $\{\mathsf{DY}^{\hbar,\mathsf{int},0}_{\mathfrak{b}_B}\}_{B\subseteq D}$. Moreover, the representations of $B^{\underline{m}}_D$ corresponding to $(\mathcal{C},S_i)$ and $(\mathcal{C},S_i)_{(u,F)}$ are equivalent, *i.e.*, $(\lambda_{(u,F)})_{\mathcal{F}}=\mathsf{Ad}(u_{\mathcal{F}})\circ\lambda_{\mathcal{F}}$.

15.13. **The local monodromies S_i.** Let $i\in D$ be a fixed vertex, and set $\overline{\mathfrak{h}}_i=\mathsf{k}\alpha_i^\vee\oplus\mathsf{k}\lambda_i^\vee$.

Lemma. *Let $S^{(1)}_i,S^{(2)}_i$ be two elements of the form (15.3) which satisfy the relation*

$$J(S^{(k)}_i)_{12}J^{-1}=J\cdot R_{21}\cdot J_{21}^{-1}(S^{(k)}_i)_1(S^{(k)}_i)_2 \tag{15.4}$$

for some $R,J\in 1^{\otimes 2}+\hbar\cdot((U\overline{\mathfrak{g}}_i)^{\otimes 2})^{\overline{\mathfrak{h}}_i}[\![\hbar]\!]$. Then, there exist unique $u,v\in\mathsf{k}[\![\hbar]\!]$ such that

$$S^{(2)}_i=\mathsf{Ad}(e^{u\alpha_i^\vee+v\lambda_i^\vee})(S^{(1)}_i)$$

[22] A braided pre–Coxeter structure on $\{\mathsf{DY}^{\hbar}_{\mathfrak{b}_B}\}_{B\subseteq D}$ is *weight–zero* if it is defined over the category $\mathsf{DY}^{\hbar}_{\mathfrak{h}}$.

PROOF. Let $S_i^{(k)} = \widetilde{s}\cdot \mathsf{p}(\underline{S}_i^{(k)})$, where

$$\mathsf{p}(\underline{S}_i^{(k)}) = 1 + \sum_{N\geqslant 0} \hbar^n s_n^{(k)} \qquad s_n^{(k)} \in (U\overline{\mathfrak{g}}_i)^{\overline{\mathfrak{h}}}$$

The identity above reads

$$\mathsf{p}(\underline{S}_i^{(k)})_{12}\cdot J^{-1} = R_{21}^{\theta}\cdot (J_{21}^{-1})^{\theta}\cdot \mathsf{p}(\underline{S}_i^{(k)})_1\cdot \mathsf{p}(\underline{S}_i^{(k)})_2 \tag{15.5}$$

where θ is the Chevalley involution, acting as -1 on $\overline{\mathfrak{h}}_i$.

We construct two sequences

$$u_n = \sum_{k=0}^{n} a_k\hbar^n \qquad \text{and} \qquad v_n = \sum_{k=0}^{n} b_k\hbar^n \qquad a_k, b_k \in \mathsf{k}$$

such that

$$S_i^{(2)} = e^{u_n\alpha_i^\vee + v_n\lambda_i^\vee} S_i^{(1)} e^{-u_n\alpha_i^\vee - v_n\lambda_i^\vee} \qquad \text{mod } \hbar^{n+1} \tag{15.6}$$

Since $S_2 = S_1 = \widetilde{s}$ modulo $\hbar$, we may assume $a_0 = 0 = b_0$. Assume therefore a_k, b_k defined for $k = 0, 1, \dots, n$ for some $n \geqslant 0$. Let $(S_i^{(1)})'$ be given by the right–hand side of (15.6), so that

$$\mathsf{p}(\underline{S}_i^{(2)}) = \mathsf{p}(\underline{S}_i^{(1)})' + \hbar^{n+1}\eta \qquad \text{mod } \hbar^{n+2}$$

for some $\eta \in (U\overline{\mathfrak{g}}_i)^{\overline{\mathfrak{h}}_i}$. One readily checks that $\mathsf{p}(\underline{S}_i^{(1)})'$ satisfies (15.5), since $e^{u_n\alpha_i^\vee + v_n\lambda_i^\vee}$ is group–like element in $U\overline{\mathfrak{h}}_i[\![\hbar]\!]$. Subtracting from this the coproduct identity for $\mathsf{p}(\underline{S}_i^{(2)})$, and computing modulo $\hbar^{n+2}$, we find that

$$d_H(\eta) = \eta_2 - (\eta)_{12} + \eta_1 = 0$$

Therefore, η is a primitive element in $(U\overline{\mathfrak{g}}_i)^{\overline{\mathfrak{h}}_i}$. It follows $\eta = c\cdot\alpha_i^\vee + d\cdot\lambda_i^\vee$, for some $c, d \in \mathsf{k}$. Then for $a_{n+1} = -c/2$, $b_{n+1} = -d/2$, we get

$$e^{(a_{n+1}\alpha_i^\vee + b_{n+1}\lambda_i^\vee)\hbar^{n+1}} (S_i^{(1)})' e^{-(a_{n+1}\alpha_i^\vee + b_{n+1}\lambda_i^\vee)\hbar^{n+1}} = (S_i^{(1)})' e^{(c\alpha_i^\vee + d\lambda_i^\vee)\hbar^{n+1}} = \underline{S}_i^{(2)}$$

modulo $\hbar^{n+2}$. By induction, one gets $u, v \in \mathsf{k}[\![\hbar]\!]$ such that

$$S_i^{(2)} = \mathsf{Ad}(e^{u\alpha_i^\vee + v\lambda_i^\vee})(S_i^{(1)})$$

□

Since the coproduct identity (14.2) has the form (15.4), where $R = R_i \in \widehat{\mathcal{U}}^2_{\{i\},0}$ is an R–matrix and $J = J_i \in \widehat{\mathcal{U}}^2_{\{i\},0}$ is a twist, we get the following

Corollary. *Up to gauge transformation, a good (resp. universal) braided pre–Coxeter structure on* $\{\mathsf{DY}^{\hbar,\mathsf{int},0}_{\overline{\mathfrak{b}}_B}\}_{B\subseteq D}$ *can be completed to at most one universal braided Coxeter structure.*

15.14. **Coxeter structures on extended Kac–Moody algebras.** Let $\overline{\mathfrak{g}}$ be an extended symmetrisable Kac–Moody algebra with negative Borel subalgebra $\overline{\mathfrak{b}}$ and Dynkin diagram D, and $\mathsf{DY}^{\hbar,\mathsf{int},0}_{\overline{\mathfrak{b}}}$ the deformation category of integrable Drinfeld–Yetter $\overline{\mathfrak{b}}$–modules.

The following is the main result of this paper.

Theorem. *Let $k = 1,2$, and*

$$\mathcal{C}_k = (\Phi_B^{(k)}, J_{\mathcal{F}}^{(k)}, \Upsilon_{\mathcal{FG}}^{(k)}, S_i^{(k)})$$

two universal braided Coxeter structures on $\{\mathsf{DY}_{\overline{\mathfrak{b}}_B}^{\hbar,\mathsf{int},0}\}_{B\subseteq D}$ corresponding to a fixed labeling $\underline{m}$ on D. Then,

(1) *There is a twist (u,F) such that $\mathcal{C}_2 = (\mathcal{C}_1)_{(u,F)}$.*
(2) *The twist (u,F) is unique up to a unique gauge a.*

Proof. Let $(\mathcal{C}_k, \{S_i^{(k)}\})$, $k = 1,2$, be two universal Coxeter structures on $\mathsf{DY}_{\overline{\mathfrak{b}}}^{\mathsf{int}}$. By 13.4, there is a universal twisting (u,F) such that

$$\mathcal{C}_2 = (\mathcal{C}_1)_{(u,F)}$$

where u is uniquely determined, and F is uniquely determined up to multiplication with elements of the form $(a_B)_1^{-1}(a_B)_2^{-1}(a_B)_{12}$, where a_B belongs to the center of $\widehat{\mathcal{U}}_{\overline{\mathfrak{b}}}^{n}$. Therefore, $S_i^{(2)}$ and $(S_i^{(1)})_a$ are two Coxeter extensions of $\mathcal{C}_2$. By Lemma 15.13, there is a unique tuple $\underline{v} = (v_1, \dots, v_n, v_1', \dots, v_n')$, $v_i, v_i' \in \mathsf{k}[\![\hbar]\!]$, such that

$$\mathsf{Ad}(e^{v_i\alpha_i^\vee + v_i'\lambda_i^\vee})(S_i^{(1)})_u = S_i^{(2)}$$

and

$$(\mathcal{C}_2, \{S_i^{(2)}\}) = (\mathcal{C}_1, \{S_i^{(1)}\})_{(\underline{v}\circ u, F)}$$

The theorem is proved. □

Let $\mathcal{O}_{\infty,\overline{\mathfrak{g}}}^{\hbar,\mathsf{int}}$ be the category of deformation, integrable, category $\mathcal{O}_\infty$ $\overline{\mathfrak{g}}$–modules. From 15.10 and 15.11, we get the following

Corollary. *Any two universal braided Coxeter structures on $\{\mathcal{O}_{\infty,\overline{\mathfrak{g}}_B}^{\hbar,\mathsf{int}}\}_{B\subseteq D}$ are twist equivalent, with respect to a universal twist, which is unique up to a unique gauge.*

Remark. Since the labeling of the diagram D plays no role in the proof of the rigidity of braided Coxeter structures, the latter yields the following strengthening of Theorem 15.14. If $\mathcal{C}_1, \mathcal{C}_2$ are two universal braided Coxeter structures on $\mathsf{DY}_{\overline{\mathfrak{b}}}^{\hbar,\mathsf{int},0}$ corresponding to the labelings $\{m_{ij}^1\}$, $\{m_{ij}^2\}$, then there is a twist (u,F) such that $\mathcal{C}_2 = (\mathcal{C}_1)_{(u,F)}$, which is unique up to a unique gauge. In particular, the local monodromies of $\mathcal{C}_1, \mathcal{C}_2$ satisfy the braid relations with respect to the labeling $\{\min(m_{ij}^1, m_{ij}^2)\}$.

15.15. **Coxeter structures on diagrammatic Kac–Moody algebras.** We mention in Remark 15.2 that the definition of extended Kac–Moody algebra is prompted by the fact that not all Kac–Moody algebras are diagrammatic and, more specifically, not all symmetrisable Kac–Moody algebras are graded, as Lie bialgebras, over the diagrammatic semigroup $\mathbb{S}$ associated to their root system (cf. 12.8). Nonetheless, one observes easily that a large class of (non–extended) symmetrisable Kac–Moody algebras are $\mathbb{S}$–graded, including those of finite, affine, and hyperbolic type. In [2, 11], we refer to these as *Cartan diagrammatic* symmetrisable Kac–Moody algebras. It is evident that the results described above hold verbatim for these Lie bialgebras. Therefore, we get the following

Theorem. *Let $\mathfrak{g}$ be a Cartan diagrammatic symmetrisable Kac–Moody algebra (in particular, of finite, affine, or hyperbolic type). Any two universal braided Coxeter structures on $\{\mathcal{O}_{\infty,\mathfrak{g}_B}^{\hbar,\mathsf{int}}\}_{B\subseteq D}$ are twist equivalent, with respect to a universal twist, which is unique up to a unique gauge.*

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